On the chaotic behavior of non-flat billiards*

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Abstract

We study the problem of the motion of a particle on a non-flat billiard. The particle is subject to the gravity and to a small amplitude periodic (or almost periodic) forcing and is reflected with respect to the normal axis when it hits the boundary of the billiard. We prove that the unperturbed problem has an impact homoclinic orbit and give a Melnikov type condition so that the perturbed problem exhibit chaotic behavior in the sense of Smale's horseshoe.

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1 Introduction

Impact conditions naturally appear in several interesting mechanical systems. For example an inverted pendulum impacting on rigid walls under external periodic excitation is studied in [8], a Duffing vibro-impact oscillator in [16] and other interesting impact models emerge from understanding the dynamics of rigid blocks [11, 14]. Many more stimulating examples of impact oscillators are given in books [3,4,6,9,12,13] where different numerical and analytical methods are described to study their dynamics.

Besides the above examples, there is a broad variety of impact systems represented by billiards. A billiard is essentially given by a convex domain $\Omega \subset \mathbb{R}^2$ with piecewise smooth boundary and a particle on it whose motion follows the usual Newton laws of dynamics until it reaches the

* *This paper is dedicated to Judita whose light has been suddenly switched off at the early age of 23 years*

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boundary of Ω at which points it is reflected in the opposite direction with respect to the normal to the boundary at that point, keeping the same scalar velocity. Of course we only consider trajectories hitting the boundary of Ω at its regular points. The theory of flat billiards is by now classical and very well developed. We refer the reader to [5] for more details and references. However, other kinds of billiards are also studied. According to [10], for example, a billiard in a broad sense is the geodesic flow on a Riemannian manifold with boundary.

In this paper we consider such a different kind of billiards: the dynamics of the particle evolves on a surface in $\mathbb{R}^3$, it has unitary mass and is subject to the gravity and an almost periodic forcing alone. Of course, as on alternative view, such a dynamics may also model a particle moving on a flat billiard immersed in a magnetic field.

The surface is described by a graph $z = f(x, y)$ of a function $f \in C^5\left(\mathbb{R}^2, \mathbb{R}\right)$, $f(x, y) \geq 0$ and $(x, y) \in \overline{\Omega}$. The particle is forced to remain in the surface in the sense that, each time it hits the boundary of $\mathcal{S} := \{(x, y, z) \mid (x, y) \in \Omega, z = f(x, y)\}$ it is reflected in the opposite direction with respect to the normal. By normal here we mean a vector $\vec{n}$ in the tangent plane to $\mathcal{S}$ which is orthogonal to the tangent vector to $\partial\mathcal{S}$ at the point of $\partial\mathcal{S}$. To be more precise, suppose $(x(s), y(s), f(x(s), y(s)))$ is a parametric representation of $\partial\mathcal{S}$ then $\vec{n}$ is orthogonal to the tangent vector to $\partial\mathcal{S}$: $\vec{T} = (x'(s), y'(s), x'(s)f_x(x(s), y(s)) + y'(s)f_y(x(s), y(s)))$ and to the normal vector to the surface $z = f(x, y)$: $\vec{B} = (-f_x(x(s), y(s)), -f_y(x(s), y(s)), 1)$. So $\vec{n} = \vec{B} \wedge \vec{T}$. For example if, as we assume in this paper, $f(x, y) = 0$ in a neighborhood of the boundary $\partial\Omega$, then:

$$\vec{n} = \begin{pmatrix} -y'(s) \\ x'(s) \\ 0 \end{pmatrix}.$$

Using D'Alembert principle the equation of motion of the particle without an almost periodic forcing, in $\mathcal{S} \setminus \partial\mathcal{S}$ is given by [7, p. 662]

$$\begin{aligned} \ddot{x} &= \lambda f_x(x, y) \\ \ddot{y} &= \lambda f_y(x, y) \\ \ddot{z} &= -\lambda - g \end{aligned} \tag{1.1}$$

where g is the gravitation constant. The constraint $z(t) = f(x(t), y(t))$ and equation (1.1) give

$$-\lambda - g = \ddot{z} = \lambda f_x(x, y)^2 + f_{xx}\dot{x}^2 + 2f_{xy}(x, y)\dot{x}\dot{y} + f_{yy}(x, y)\dot{y}^2 + \lambda f_y(x, y)^2,$$

which implies

$$\lambda = -\frac{g + \langle H_f(x, y)\left(\begin{smallmatrix}\dot{x}\\ \dot{y}\end{smallmatrix}\right), \left(\begin{smallmatrix}\dot{x}\\ \dot{y}\end{smallmatrix}\right)\rangle}{1 + \|\nabla f\|^2}, \tag{1.2}$$

where ∇f and H_f is the gradient and Hessian of f, respectively. We note that $\nabla f = f'^*$, which we use several times in our paper. As a consequence the problem is reduced to study the behaviour of solutions of an almost periodic perturbation of the following unperturbed differential equation on $\Omega = \{(x, y) \mid x \geq 0, 0 \leq y \leq x \tan\beta\}$:

$$\begin{aligned} \ddot{x} &= \lambda f_x(x, y) \\ \ddot{y} &= \lambda f_y(x, y) \end{aligned} \tag{1.3}$$

where $\lambda = \lambda(x, y, \dot{x}, \dot{y})$ is as in (1.2), and $z = f(x, y)$, together with the requirement that, when $(x(t), y(t)) \in \partial\Omega$ then $(\dot{x}(t), \dot{y}(t))$ is reflected with respect to the normal to $\partial\Omega$ at $(x(t), y(t))$. Hence the solution of (1.1) is forced to remain in Ω.

We emphasize that the main purpose of this paper is to introduce a new class of impact systems modeled by nonlinear billiards with chaotic behaviour. So instead of a gravitational force, we could consider other force fields acting on the particle under which it is moving inside Ω. In this paper, we consider the gravitational field since we think that this problem is interesting itself and in addition, it is rather sophisticated for showing all difficulties of technical computations and theoretical background.

To continue, given the messy nature of equation (1.2), we assume

$$f(x,y) = F\left((x-a)^2 + (y-b)^2\right) \tag{1.4}$$

with $a > 0$, $0 < b < a\tan\beta$, $0 < \beta < \frac{\pi}{2}$, and F is a C^5 function in $[0,\infty)$ whose support is contained in an interval $[0, r_0^2]$ with $r_0 > 0$ sufficiently small that the closed ball $\overline{B((a,b),r_0)}$ is contained in $\overset{\circ}{\Omega}$ and such that $F' \leq 0$ with $F'(0) < 0$.

Example 1.1. For illustration, as a concrete example, we take $a = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$, $b = \sin\frac{\pi}{6} = \frac{1}{2}$, $\beta = \frac{\pi}{3}$ and $F(r) = (1-16r)^6$ for $0 \leq r \leq r_0^2 = \frac{1}{16}$ and $F(r) = 0$ for $r \geq \frac{1}{16}$.

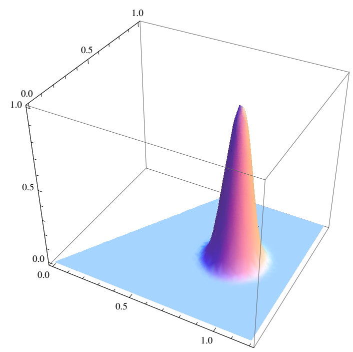

Figure 1: The graph of $f(x,y)$ in this concrete case on $0 \leq x \leq 1.2$ and $0 \leq y \leq \min\{\sqrt{3}x, 1\}$.

The plan of this paper is as follows. In Section 2 we will prove that, if r_0 is sufficiently small then equation (1.3) has an impact homoclinic solution that satisfies assumptions $(H1)-(H3)$ in [2]. Then in Section 3 we construct the Melnikov function associated to the almost periodic perturbation. Section 4 contains our main result that is obtained by an application of [2, Theorem 4.2]. Finally, in Section 5, we give another method for computing the Melnikov function that does not make use of first integrals of unperturbed system.

Certainly our method can be extended to other kind of domains Ω as, for example, triangles or other convex subsets of $\mathbb{R}^2$ and to homoclinic solutions of the unperturbed systems with more impacts. However, for sake of simplicity, here we study homoclinic solutions with two impacts (see Figure 3).

2 Homoclinic impact solutions

Passing to polar coordinates around the point (a,b), i.e. taking

$$x = a + \rho\cos\varphi, \quad y = b + \rho\sin\varphi \tag{2.1}$$

we get $f(x,y) = F(\rho^2)$ and then

$$\begin{aligned} \nabla f(x,y) &= 2F'(\rho^2)\begin{pmatrix} x-a \\ y-b \end{pmatrix}, \\ H_f(x,y) &= 2F'(\rho^2)\mathbb{I} + 4F''(\rho^2)\begin{pmatrix} (x-a)^2 & (x-a)(y-b) \\ (x-a)(y-b) & (y-b)^2 \end{pmatrix} \end{aligned} \tag{2.2}$$

that is

$$f_x(x,y) = 2F'\left(\rho^2\right)\rho\cos\varphi, \quad f_y(x,y) = 2F'\left(\rho^2\right)\rho\sin\varphi$$
$$f_{xx}(x,y) = 2F'\left(\rho^2\right) + 4F''\left(\rho^2\right)\rho^2\cos^2\varphi,$$
$$f_{xy}(x,y) = 4F''\left(\rho^2\right)\rho^2\cos\varphi\sin\varphi,$$
$$f_{yy}(x,y) = 2F'\left(\rho^2\right) + 4F''\left(\rho^2\right)\rho^2\sin^2\varphi.$$

From (1.2) it follows

$$\lambda = \lambda\left(\rho^2,\dot\rho^2,\dot\varphi^2\right) := -\frac{g + 2F'\left(\rho^2\right)\left(\rho^2\dot\varphi^2 + \dot\rho^2\right) + 4\rho^2\dot\rho^2 F''\left(\rho^2\right)}{1 + 4\rho^2 F'\left(\rho^2\right)^2} \tag{2.3}$$

and (1.1) reads

$$\begin{gathered} -\sin\varphi\left(2\dot\varphi\dot\rho + \rho\ddot\varphi\right) + \cos\varphi\left(-\rho\dot\varphi^2 + \ddot\rho\right) = 2\lambda\left(\rho^2,\dot\rho^2,\dot\varphi^2\right)F'\left(\rho^2\right)\rho\cos\varphi \\ \cos\varphi\left(2\dot\varphi\dot\rho + \rho\ddot\varphi\right) + \sin\varphi\left(-\rho\dot\varphi^2 + \ddot\rho\right) = 2\lambda\left(\rho^2,\dot\rho^2,\dot\varphi^2\right)F'\left(\rho^2\right)\rho\sin\varphi. \end{gathered}$$

Since these equations holds for any value of (ρ,φ) we get, equating coefficients of $\cos\varphi$ and $\sin\varphi$:

$$\begin{cases} \ddot\rho - \rho\dot\varphi^2 = 2\lambda\left(\rho^2,\dot\rho^2,\dot\varphi^2\right)F'\left(\rho^2\right)\rho \\ 2\dot\varphi\dot\rho + \rho\ddot\varphi = 0. \end{cases} \tag{2.4}$$

Clearly the second equation in (2.4) is equivalent to

$$\dot\varphi\rho^2 = c \tag{2.5}$$

for a constant $c \in \mathbb{R}$. Since we are looking for a solution such that $\lim_{t\to\pm\infty}\rho(t)^2 + \dot\rho(t)^2 = 0$ and $\rho(t) > 0$ we see that $c = 0$ in (2.5) and hence $\dot\varphi(t) = 0$. So (2.4) has the form

$$\ddot\rho = 2\lambda\left(\rho^2,\dot\rho^2,0\right)F'\left(\rho^2\right)\rho \tag{2.6}$$

and $\dot\varphi = 0$ (i.e. $\varphi =$ constant). Hence the projection onto the (x,y) plane of impact homoclinic solutions of (1.1) evolve along straight lines in some intervals such as $]-\infty, t_1^*]$ and $[t_2^*,\infty[$. Changing t with $-t$, if needed, we may assume that t_1^* and t_2^* are such that $y(t_1^*) = 0$ and $(x(t),y(t)) \in \Omega$ for any $t < t_1^*$ and $y(t_2^*) = x(t_2^*)\tan\beta = 0$ and $(x(t),y(t)) \in \Omega$ for any $t > t_2^*$. So for either $t < t_1^*$ or $t > t_2^*$ we have

$$\begin{aligned} x(t) &= a + \rho(t)\cos\varphi, \\ y(t) &= b + \rho(t)\sin\varphi \end{aligned}$$

where φ is constant and $\rho(t)$ is a solution of (2.6) such that $\lim_{t\to\pm\infty}\rho(t) = 0$. Moreover at $t = t_1^*$ or $t = t_2^*$ the solution is *reflected* so that $\dot y((t_1^*)^+) = -\dot y((t_1^*)^-)$ and

$$\begin{pmatrix} \dot x((t_2^*)^+) \\ \dot y((t_2^*)^+) \end{pmatrix} = M_\beta \begin{pmatrix} \dot x((t_2^*)^-) \\ \dot y((t_2^*)^-) \end{pmatrix}$$

and M_β is the matrix of the reflection with respect to the line $y = x\tan\beta$ i.e.:

$$M_\beta = \begin{pmatrix} \cos 2\beta & \sin 2\beta \\ \sin 2\beta & -\cos 2\beta \end{pmatrix}.$$

As a first step we look then for solutions of (2.6) such that $\lim_{t\to\infty}\rho(t) = 0$. Setting $r = \rho^2$ and $w = \dot\rho^2$ (2.6) reads

$$\begin{aligned} \dot r &= 2\rho\dot\rho \\ \dot w &= 4\rho\dot\rho\lambda(r,w,0)F'(r) \end{aligned}$$

so that $\frac{dw}{dr} = 2\lambda(r, w, 0)F'(r)$. Hence we look for solutions of (2.6) of the form $\dot\rho^2 = w(\rho^2)$ with $w(0) = 0$. We have seen that such a function $w(r)$ satisfies

$$w'(r) = 2\lambda\left(r, w(r), 0\right) F'(r) = -4\frac{F'(r) + 2rF''(r)}{1 + 4rF'(r)^2} F'(r)w(r) - \frac{2gF'(r)}{1 + 4rF'(r)^2}$$

together with $w(0) = 0$. Since $\frac{d}{dr} \log \frac{1}{1+4rF'(r)^2} = -4\frac{F'(r)+2rF''(r)}{1+4rF'(r)^2}F'(r)$, we obtain

$$w(r) = \frac{2g\left(F(0) - F(r)\right)}{1 + 4rF'(r)^2}$$

or

$$\dot\rho^2 = \frac{2g\left(F(0) - F(\rho^2)\right)}{1 + 4\rho^2F'(\rho^2)^2}.$$

Since $F' \le 0$ with $F'(0) < 0$, we get $F(0) - F(r) > 0$ for any $r > 0$ and so $\dot\rho \ne 0$ for any $t \in [t_2^*, \infty[$. As a consequence $\dot\rho(t) < 0$ for any t in the same interval and

$$\dot\rho = -\sqrt{\frac{2g\left(F(0) - F(\rho^2)\right)}{1 + 4\rho^2F'(\rho^2)^2}}.$$

Then we obtain the following equation for $r = \rho^2$:

$$\dot r = -\sqrt{\frac{8gr(F(0) - F(r))}{1 + 4rF'(r)^2}} = -r\sqrt{\frac{8g\mathcal{H}(r)}{1 + 4rF'(r)^2}} \tag{2.7}$$

where $F(0) - F(r) = r\mathcal{H}(r)$, $\mathcal{H} \in C^4\left([0, \infty), (0, \infty)\right)$. Since $F(r) < F(0)$ for any $r > 0$ we see that equation (2.7) does not have fixed points in $\{r > 0\}$ and $\dot r < 0$. So a solution of (2.7) starting from a given $r^* > 0$ is defined on $[0, \infty)$ and its $\omega-$limit set is $r = 0$. Next $u(t) = r(-t)$ satisfies

$$\dot u = u\sqrt{\frac{8g\mathcal{H}(u)}{1 + 4uF'(u)^2}} \tag{2.8}$$

and then $r(-t)$ is strictly increasing and positive. As a consequence at a certain time t_0 it will result $r(-t_0) = r_0^2$ but then for $t > t_0$, $u(t) = r(-t)$ satisfies:

$$\dot u = \sqrt{8gF(0)u}$$

with $u(t_0) = r_0^2$. Hence $r(-t)$ exists for any $t > 0$ and satisfies $\lim\limits_{t\to\infty} r(-t) = +\infty$. As a consequence the $\alpha-$limit set of any solution of (2.7) starting from a positive r^* is $+\infty$. So, any solution of (2.7) starting from $r^* > 0$ is strictly decreasing and satisfies:

$$\lim_{t\to-\infty} r(t) = \infty;$$
$$\lim_{t\to+\infty} r(t) = 0.$$

Let $r_0(t)$ be any (fixed) positive solution of (2.7). Then any other positive solution of equation (2.7) is obtained from $r_0(t)$ by a time shift $r_1(t) = r_0(t+t_1)$, t_1 satisfying $r_0(t_1) = r_1(0)$. Moreover, $r_0(-t)$ solves equation (2.8). We set

$$\rho_0(t) = \sqrt{r_0(t)}.$$

$\rho_0(t)$ is a decreasing positive function such that

$$\lim_{t\to\infty} \rho_0(t) = 0, \qquad \lim_{t\to-\infty} \rho_0(t) = +\infty$$

hence $\rho_0(t)$ takes all positive values once when t varies in $\mathbb{R}$ and for any given $(x, y) \in \overline{\Omega} \setminus \{(a, b)\}$, there exists a unique solution $t = t_{x,y}$ of

$$\rho_0(t_{x,y}) = \sqrt{(x-a)^2 + (y-b)^2}$$

So we have stable and unstable solutions:

$$\gamma_s(t, x, y) = (a, b) + \frac{\rho_0(t + t_{x,y})}{\sqrt{(x-a)^2 + (y-b)^2}}(x - a, y - b) \quad t \geq 0 \tag{2.9}$$

and

$$\gamma_u(t, x, y) = (a, b) + \frac{\rho_0(t_{x,y} - t)}{\sqrt{(x-a)^2 + (y-b)^2}}(x - a, y - b) \quad t \leq 0 \tag{2.10}$$

of equation (1.3) with $\gamma_{s,u}(0, x, y) = (x, y)$.

We recall that we have chosen F so that its support is a subset of the interval $[0, r_0^2]$ where $r_0 > 0$ is chosen so that the closed ball with radius r_0 and center at (a, b) is contained in the interior of Ω. As a consequence the dynamics in $\Omega \setminus B((a, b), r_0)$ (where $F(\rho^2) = 0$) is governed by the equation:

$$\begin{aligned} &\ddot{\rho} - \rho\dot{\varphi}^2 = 0 \\ &2\dot{\varphi}\dot{\rho} + \rho\ddot{\varphi} = 0 \end{aligned}$$

that on account of (2.1), is equivalent to

$$\ddot{x} = 0, \qquad \ddot{y} = 0.$$

So, if $(x_0, y_0, \dot{x}_0, \dot{y}_0)$ are the initial values of the solution we have

$$(x(t) - x_0)\dot{y}_0 = (y(t) - y_0)\dot{x}_0$$

that is: in $\Omega \setminus B((a, b), r_0)$ solutions evolve along straight lines.

Summarizing we have seen that, if the support of $f(x, y) := F(\rho^2)$, $\rho^2 = (x-a)^2 + (y-b)^2$), is contained in the ball $B := B((a, b), r_0)$ with r_0 sufficiently small, then impact homoclinic solutions evolve on straight lines. Impact homoclinic orbits are then constructed as follows. We draw a straight line from (a, b) until it reaches the $x-$axis $y = 0$. From the intersection point we start a new straight line in Ω making a symmetric angle with the normal to $y = 0$ at this intersection point, until it reaches the line $y = x \tan \beta$ where we repeat the procedure. We obtain a homoclinic orbit if and only if this last straight line passes through the fixed point (a, b). Because of the symmetry this condition is equivalent to the following geometric construction. We take the symmetric points

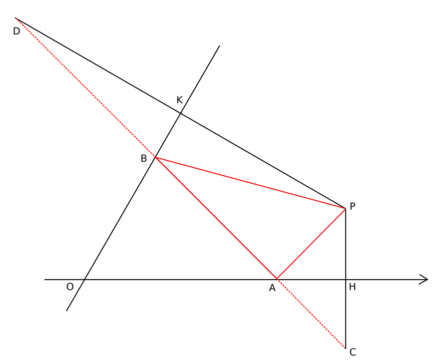

Figure 2: The impact homoclinic orbit (in solid red) and its construction using symmetries

of $P = (a, b)$ with respect to the lines $y = 0$ and $y = x \tan \beta$ and join them with a straight line (see Figure 1). The impact points of the homoclinic orbit are the intersections A and B of these lines with the lines $y=0$ and $y=x\tan\beta$ and the orbit of the homoclinic solution is represented by the triangle whose vertex are A, B and the fixed point (a,b). Note that in order that this is the homoclinic orbit we also need than the closure of the ball $B((a,b),r_0)$ should not intersect the segment AB.

This construction should convince the reader of the uniqueness of the impact homoclinic orbit. Moreover it makes it clear why we need to take the edges of the billiard non orthogonal. In fact, if the impact lines are orthogonal, the line through the symmetric points of (a,b) with respect to the orthogonal axis passes through the origin and then we do not have impacts. Even worse is the case $\frac{\pi}{2}<\beta<\pi$ as the reader can easily verify. Since solutions of (1.3) that belong to a small neighborhood of the homoclinic orbit take approximately the same time as the homoclinic orbit between the lines $y=0$ and $y=x\tan\beta$ we will forget this branch of the solutions and restrict our attention to the branches tending to the fixed point. So we define a map R from the manifold $\{(x,0,u,v)\mid x>0,u<0,v<0\}$ (corresponding to the manifold of intersection points of solutions of (1.3) with the line $y=0$) into the manifold $\{(x,x\tan\beta,u,v)\}$ that takes the above reflections and translation into account. This map is the combination of the following: first we reflect the velocity with respect to the axis $y=0$, then we move linearly along this direction until we reach the manifold $y=x\tan\beta$ where we take the reflection with respect to $x+y\tan\beta=0$. Summarizing, starting from the point $(x,0,u,v)$ with $x>0$, $u<0$, $v<0$, first we take the first reflection to get the point $(x,0,u,-v)$; next we consider the intersection of the half-line $(x+su,-sv)$, $s\geq 0$ with the half-line $y=x\tan\beta$, $x>0$. Thus we solve $(x+su)\tan\beta=-sv$ to get $s=\frac{-x\tan\beta}{v+u\tan\beta}$, and the intersection point is

$$\left(\frac{xv}{v+u\tan\beta},\frac{xv\tan\beta}{v+u\tan\beta}\right).$$

Since there is no force acting on the ball during this part of the trajectory the particle reaches the intersection point $\left(\frac{xv}{v+u\tan\beta},\frac{xv\tan\beta}{v+u\tan\beta}\right)$ with speed $(u,-v)$. So the velocity of the particle after the reflection along the line $y=x\tan\beta$, is

$$M_\beta\begin{pmatrix}u\\-v\end{pmatrix}=\begin{pmatrix}u\cos 2\beta-v\sin 2\beta\\ v\cos 2\beta+u\sin 2\beta\end{pmatrix}.$$

Summarizing, we get

$$R(x,0,u,v)=\left(\frac{xv}{v+u\tan\beta},\frac{xv\tan\beta}{v+u\tan\beta},u\cos 2\beta-v\sin 2\beta,v\cos 2\beta+u\sin 2\beta\right)$$

for $x>0$, $u<0$, $v<0$. Now we look for such an array starting from (a,b) which after the above reflections passes true this point (a,b). As we have seen the reflecting points of such an orbit are obtained as follows: we reflect (a,b) with respect to $y=0$ to get $P_1:=(a,-b)$ and also with respect $y=\tan\beta x$, to get $P_2:=(a\cos 2\beta+b\sin 2\beta,-b\cos 2\beta+a\sin 2\beta)$. The reflecting points are the intersection of the lines $y=0$ and $y=x\tan\beta$ with the segment connecting P_1 and P_2:

$$S_{a,b}:=\Big\{s(a,-b)+(1-s)\left(a\cos 2\beta+b\sin 2\beta,-b\cos 2\beta+a\sin 2\beta\right)\mid s\in[0,1]\Big\}.$$

The first intersection point is given by

$$I_1=S_{a,b}\cap\{y=0,x>0\}=\left(\frac{\left(a^2+b^2\right)\cos\beta}{a\cos\beta+b\sin\beta},0\right)\tag{2.11}$$

for $s_1=\frac{2a\cos\beta-b\cos 2\beta\csc\beta}{2a\cos\beta+2b\sin\beta}$. Using $a>b\cot\beta$ we derive $2a\cos\beta-b\cos 2\beta\csc\beta>2b\cot\beta\cos\beta-b\cos 2\beta\csc\beta=b\csc\beta>0$. Next we have $1-s_1=\frac{b\csc\beta}{2a\cos\beta+2b\sin\beta}>0$. Hence $0<s_1<1$.

The second intersection point is given by

$$I_2 = S_{a,b} \cap \{y = x\tan\beta, x > 0\} = \left(\frac{(a^2+b^2)\cos^2\beta}{a}, \frac{(a^2+b^2)\cos\beta\sin\beta}{a}\right) \tag{2.12}$$

for $s_2 = \frac{a-b\cot\beta}{2a}$. Clearly $0 < s_2 < 1$.

Next, it is elementary to see that the distance of the point (a,b) from the segment $S_{a,b}$ is $\frac{2b(a\sin\beta-b\cos\beta)}{\sqrt{a^2+b^2}}$ and from the line $y = x\tan\beta$ is $a\sin\beta - b\cos\beta$. So we suppose that

$$r_0 < \min\left\{\frac{2b(a\sin\beta-b\cos\beta)}{\sqrt{a^2+b^2}}, a\sin\beta - b\cos\beta, b\right\}. \tag{2.13}$$

In Example 1.1, we have $r_0 = \frac{1}{4}$ and the above r.h.s. equals $\frac{1}{2}$.

Next shift time in such a way that $\rho_0(0) = r_0$. Then, for $t \le 0$, $\dot\rho_0(t) = -\sqrt{2gF(0)}$ and so $\rho_0(t) = r_0 - t\sqrt{2gF(0)}$. Now, our impact homoclinic orbit must be of the form:

$$\gamma(t) = \begin{cases} \gamma_u(t,x,y) & \text{for } t < 0 \\ \gamma_s(t,x,y) & \text{for } t \ge 0 \end{cases}$$

where (x,y) has to be chosen so that

$$\gamma_u(0,x,y) = I_1 \quad \text{and} \quad \gamma_s(0,x,y) = I_2.$$

The first equality is equivalent to solve, for t, the equation

$$\rho_0(t) = \|I_1 - \left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right)\| = b\frac{\sqrt{a^2+b^2}}{a\cos\beta + b\sin\beta}$$

and, similarly, the second is equivalent to solve

$$\rho_0(t) = \|I_2 - \left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right)\| = \frac{\sqrt{a^2+b^2}(a\sin\beta - b\cos\beta)}{a}$$

Since $\|I_1 - \left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right)\| > r_0$ and $\|I_2 - \left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right)\| > r_0$, the two equations have, respectively, the solutions:

$$t^-_{(a,b)} = \frac{1}{\sqrt{2gF(0)}}\left[r_0 - \frac{b\sqrt{a^2+b^2}}{a\cos\beta + b\sin\beta}\right] < 0$$

since $\frac{b\sqrt{a^2+b^2}}{a\cos\beta+b\sin\beta} > b$, and

$$t^+_{(a,b)} = \frac{1}{\sqrt{2gF(0)}}\left[r_0 - \frac{\sqrt{a^2+b^2}}{a}(a\sin\beta - b\cos\beta)\right] < 0$$

since $\frac{\sqrt{a^2+b^2}}{a}(a\sin\beta - b\cos\beta) > a\sin\beta - b\cos\beta$. So the impact homoclinic solution is:

$$\gamma(t) = \begin{cases} \gamma_-(t) := \left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right) - \frac{\rho_0(t^-_{(a,b)}-t)}{\sqrt{a^2+b^2}}\left(\begin{smallmatrix} a\sin\beta - b\cos\beta \\ a\cos\beta + b\sin\beta \end{smallmatrix}\right) & \text{for} \quad t < 0 \\ \\ \gamma_+(t) := \left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right) - \frac{\rho_0(t^+_{(a,b)}+t)}{\sqrt{a^2+b^2}}\left(\begin{smallmatrix} a\sin\beta + b\cos\beta \\ b\sin\beta - a\cos\beta \end{smallmatrix}\right) & \text{for} \quad t > 0 \end{cases} \tag{2.14}$$

In Figure 3 we draw the homoclinic orbit $(\gamma(t), f(\gamma(t))$ of (1.1) when $F(r)$ is as in Example 1.1. For such a function we have indeed $t^\pm_{(a,b)} \doteq -0.0739158$.

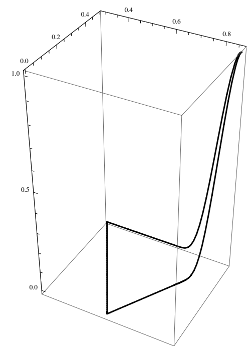

Figure 3: Impact homoclinic orbit for Figure 1.

Recall

$$\begin{aligned}
&\rho_0\left(t^-_{(a,b)}\right) = \|I_1 - (a,b)\|, \quad \rho_0\left(t^+_{(a,b)}\right) = \|I_2 - (a,b)\|,\\
&\dot\rho_0\left(t^-_{(a,b)}\right) = \dot\rho_0\left(t^+_{(a,b)}\right) = -\sqrt{2gF(0)}, \qquad (2.15)\\
&\ddot\rho_0\left(t^-_{(a,b)}\right) = \ddot\rho_0\left(t^+_{(a,b)}\right) = 0.
\end{aligned}$$

Hence

$$\begin{aligned}
\gamma_-(0) = I_1, \quad \dot\gamma_-(0) = \tfrac{\sqrt{2gF(0)}}{\sqrt{a^2+b^2}}\begin{pmatrix} b\cos\beta - a\sin\beta \\ -a\cos\beta - b\sin\beta\end{pmatrix}\\
\gamma_+(0) = I_2, \quad \dot\gamma_+(0) = \tfrac{\sqrt{2gF(0)}}{\sqrt{a^2+b^2}}\begin{pmatrix} b\cos\beta + a\sin\beta \\ -a\cos\beta + b\sin\beta\end{pmatrix}.
\end{aligned} \qquad (2.16)$$

It is easy to check that it actually holds

$$R\left(\gamma_-(0), \dot\gamma_-(0)\right) = \left(\gamma_+(0), \dot\gamma_+(0)\right).$$

Now we verify assumptions (H1)–(H3) from [2]. To this end, we rewrite (1.3) as the first order ODE

$$\begin{aligned}
\dot x_1 &= x_2\\
\dot y_1 &= y_2\\
\dot x_2 &= \lambda f_x(x_1, y_1)\\
\dot y_2 &= \lambda f_y(x_1, y_1)
\end{aligned} \qquad (2.17)$$

in $\Omega_- := \{(x_1, y_1, x_2, y_2) \mid x_1 > 0,\ 0 < y_1 < x_1 \tan\beta\}$. Then $E := (a, b, 0, 0)$ is an equilibrium of (2.17) and $(\gamma(t), \dot\gamma(t))$, where $\gamma(t)$ is the function defined in (2.14), is an impact homoclinic orbit to E of (2.17). The linearization of (2.17) in E is

$$\begin{aligned}
\dot u_1 &= u_2\\
\dot v_1 &= v_2\\
\dot u_2 &= -2gF'(0)u_1\\
\dot v_2 &= -2gF'(0)v_1.
\end{aligned}$$

Since $F'(0) < 0$ we see that E is a hyperbolic equilibrium of (2.17) with 2-dimensional stable and unstable manifolds W^s and W^u. In fact the Jacobian matrix at $(a, b, 0, 0)$ has the double eigenvalues $\mp\mu$, $\mu = \sqrt{2g|F'(0)|}$ with stable and unstable spaces given resp. by:

$$U^s = \operatorname{span}\left\{\begin{pmatrix}1\\0\\-\mu\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\-\mu\end{pmatrix}\right\} \quad \text{and} \quad U^u = \operatorname{span}\left\{\begin{pmatrix}1\\0\\\mu\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\\mu\end{pmatrix}\right\}.$$

Hence [2, (H1)] holds. Next we have $G(x_1,y_1,x_2,y_2) = y_1(y_1 - x_1\tan\beta)$ and then (we recall that $\nabla G(x_1,y_1,x_2,y_2) = G'(x_1,y_1,x_2,y_2)^*$)

$$G'(x_1,y_1,x_2,y_2) = (-y_1\tan\beta, 2y_1 - x_1\tan\beta, 0, 0)$$

from which it follows:

$$G'(\gamma_-(0),\dot\gamma_-(0)) = \left(0, -\frac{a^2+b^2}{a\cos\beta + b\sin\beta}\sin\beta, 0, 0\right).$$

So, using also (2.11), (2.12) and (2.16):

$$G'(\gamma_-(0),\dot\gamma_-(0))\begin{pmatrix} \sqrt{\frac{2gF(0)}{a^2+b^2}}(b\cos\beta - a\sin\beta) \\ -\sqrt{\frac{2gF(0)}{a^2+b^2}}(a\cos\beta + b\sin\beta) \\ \lambda f_x(I_1) \\ \lambda f_y(I_1)\end{pmatrix} = \sqrt{2gF(0)(a^2+b^2)}\sin\beta$$

Similarly we get:

$$G'(\gamma_+(0),\dot\gamma_+(0))\begin{pmatrix} \sqrt{\frac{2gF(0)}{a^2+b^2}}(b\cos\beta + a\sin\beta) \\ -\sqrt{\frac{2gF(0)}{a^2+b^2}}(a\cos\beta - b\sin\beta) \\ \lambda f_x(I_2) \\ \lambda f_y(I_2)\end{pmatrix} = -\sqrt{2gF(0)(a^2+b^2)}\sin\beta$$

which proves [2, (H2)] .

Next, we know that $(\gamma_-(0),\dot\gamma_-(0)) \in W^u$ and, from equation (2.10) we also know that:

$$(\tilde x,\tilde y,\tilde u,\tilde v) \in W^u \text{ if and only if } (\tilde x,\tilde y) = \gamma_u(t,x,y) \text{ and } (\tilde u,\tilde v) = \dot\gamma_u(t,x,y)$$

where $\gamma_u(t,x,y)$ are the functions given by equation (2.10). Of course, to describe completely the unstable manifold we do not need to let (x,y) vary in a complete neighborhood of (a,b) since we have t as parameter and $\left(\frac{x-a}{\sqrt{(x-a)^2+(y-b)^2}}, \frac{y-b}{\sqrt{(x-a)^2+(y-b)^2}}\right)$ belongs to the circle of radius 1 centered at (a,b). In other words we can describe W^u by letting t vary and taking $(u,v) = \left(\frac{x-a}{\sqrt{(x-a)^2+(y-b)^2}}, \frac{y-b}{\sqrt{(x-a)^2+(y-b)^2}}\right)$ in the circle of radius 1 around (a,b). So, choosing t^* so that $\rho_0(t^*) = 1$, W^u is described as:

$$W^u = \left\{\begin{pmatrix} a+\rho_0(t^*-t)u \\ b+\rho_0(t^*-t)v \\ -\dot\rho_0(t^*-t)u \\ -\dot\rho_0(t^*-t)v\end{pmatrix} \mid t\in\mathbb{R},\ u^2+v^2=1\right\}$$

and $(\gamma_-(0),\dot\gamma_-(0))$ corresponds to

$$t = t^* - t^-_{a,b} \qquad u = u_- := -\frac{a\sin\beta - b\cos\beta}{\sqrt{a^2+b^2}}, \qquad v = v_- := -\frac{b\sin\beta + a\cos\beta}{\sqrt{a^2+b^2}}.$$

Then, using (2.15) we obtain:

$$\mathcal{N}P_- = T_{(\gamma_-(0),\dot\gamma_-(0))}W^u = \operatorname{span}\left\{\begin{pmatrix} u_- \\ v_- \\ 0 \\ 0\end{pmatrix}, \quad \begin{pmatrix} v_- \\ -u_- \\ kv_- \\ -ku_-\end{pmatrix},\ k = \frac{\sqrt{2gF(0)}}{\|I_1 - (a,b)\|}\right\}. \tag{2.18}$$

Similarly the stable manifold W^s of the fixed point $(a,b,0,0)$ is described as:

$$W^s = \left\{\begin{pmatrix} a+\rho_0(t^*+t)u \\ b+\rho_0(t^*+t)v \\ \dot\rho_0(t^*+t)u \\ \dot\rho_0(t^*+t)v\end{pmatrix} \mid t\in\mathbb{R},\ u^2+v^2=1\right\},$$

where $(\gamma_+(0), \dot\gamma_+(0))$ corresponds to

$$t = t^+_{a,b} - t^* \qquad u = u_+ := -\frac{a\sin\beta + b\cos\beta}{\sqrt{a^2+b^2}}, \qquad v = v_+ := \frac{a\cos\beta - b\sin\beta}{\sqrt{a^2+b^2}},$$

and

$$\mathcal{R}P_+ = T_{(\gamma_+(0),\dot\gamma_+(0))}W^s = \operatorname{span}\left\{ \begin{pmatrix} u_+ \\ v_+ \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} v_+ \\ -u_+ \\ -hv_+ \\ hu_+ \end{pmatrix}, h = \frac{\sqrt{2gF(0)}}{\|I_2 - (a,b)\|} \right\}. \tag{2.19}$$

Incidentally, we note that:

$$\begin{array}{l} \gamma_-(t) = \left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right) + \rho_0(t^-_{(a,b)} - t)\left(\begin{smallmatrix} u_- \\ v_- \end{smallmatrix}\right) \\ \gamma_+(t) = \left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right) + \rho_0(t^+_{(a,b)} + t)\left(\begin{smallmatrix} u_+ \\ v_+ \end{smallmatrix}\right) \end{array} \tag{2.20}$$

Next, $\mathcal{N}G'\left(\gamma_-(0), \dot\gamma_-(0)\right) = \{e_2\}^\perp$. So (cf. [2]):

$$\mathcal{S}' := \mathcal{N}P_- \cap \mathcal{N}G'\left(\gamma_-(0), \dot\gamma_-(0)\right) = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ kv_-^2 \\ -ku_-v_- \end{pmatrix} \right\}$$

or, using $\|I_1 - \left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right)\| v_- = -b$:

$$\mathcal{S}' = \operatorname{span}\left\{ \begin{pmatrix} -\frac{b}{v_-^2} \\ 0 \\ \sqrt{2gF(0)}v_- \\ -\sqrt{2gF(0)}u_- \end{pmatrix} \right\}$$

Let $\mathcal{S}''' := DR(\gamma_-(0), \dot\gamma_-(0))\mathcal{S}'$. Then [2, (H3)] reads: $\dim[\mathcal{R}P_+ + \mathcal{S}'''] = 3$. To prove this equality we first extend R to $\mathbb{R}^4$ as follows:

$$R(x,y,u,v) = \left(\frac{xv}{v + u\tan\beta}, \frac{xv\tan\beta}{v+u\tan\beta}, u\cos 2\beta - v\sin 2\beta, v\cos 2\beta + u\sin 2\beta \right).$$

As it has been observed in [2] the way we choose such an extension does not affect the result. We get the Jacobian matrix:

$$J_R(x,y,\mu u_-, \mu v_-) = \begin{pmatrix} -\frac{\sqrt{a^2+b^2}}{a}v_-\cos\beta & 0 & -\frac{a^2+b^2}{a^2\mu}xv_-\cos\beta\sin\beta & \frac{a^2+b^2}{a^2\mu}xu_-\cos\beta\sin\beta \\ -\frac{\sqrt{a^2+b^2}}{a}v_-\sin\beta & 0 & -\frac{a^2+b^2}{a^2\mu}xv_-\sin^2\beta & \frac{a^2+b^2}{a^2\mu}xu_-\sin^2\beta \\ 0 & 0 & \cos(2\beta) & -\sin(2\beta) \\ 0 & 0 & \sin(2\beta) & \cos(2\beta) \end{pmatrix}$$

then

$$J_R(x,y,\mu u_-,\mu v_-)\begin{pmatrix} 1 \\ 0 \\ kv_-^2 \\ -ku_-v_- \end{pmatrix} = \begin{pmatrix} -\frac{a^2+b^2}{a^2}v_-\cos\beta\left(\frac{k}{\mu}x(u_-^2+v_-^2)\sin\beta + \frac{a}{\sqrt{a^2+b^2}}\right) \\ -\frac{a^2+b^2}{a^2}v_-\sin\beta\left(\frac{k}{\mu}x(u_-^2+v_-^2)\sin\beta + \frac{a}{\sqrt{a^2+b^2}}\right) \\ kv_-(v_-\cos(2\beta) + u_-\sin(2\beta)) \\ kv_-(v_-\sin(2\beta) - u_-\cos(2\beta)) \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{a^2+b^2}{a^2}v_-\cos\beta\left(\frac{k}{\mu}x\sin\beta + \frac{a}{\sqrt{a^2+b^2}}\right) \\ -\frac{a^2+b^2}{a^2}v_-\sin\beta\left(\frac{k}{\mu}x\sin\beta + \frac{a}{\sqrt{a^2+b^2}}\right) \\ -kv_-\frac{a\cos\beta - b\sin\beta}{\sqrt{a^2+b^2}} \\ -kv_-\frac{a\sin\beta+b\cos\beta}{\sqrt{a^2+b^2}} \end{pmatrix} = v_-\begin{pmatrix} -\frac{a^2+b^2}{a^2}\cos\beta\left(\frac{k}{\mu}x\sin\beta + \frac{a}{\sqrt{a^2+b^2}}\right) \\ -\frac{a^2+b^2}{a^2}\sin\beta\left(\frac{k}{\mu}x\sin\beta + \frac{a}{\sqrt{a^2+b^2}}\right) \\ -kv_+ \\ ku_+ \end{pmatrix}.$$

With $\mu=\sqrt{2gF(0)}$, $x=\frac{(a^2+b^2)\cos\beta}{a\cos\beta+b\sin\beta}$ and $k=\frac{\sqrt{2gF(0)}}{-b}v_-$ we get $\frac{k}{\mu}x=\frac{\sqrt{a^2+b^2}}{b}\cos\beta$ and then:

$$J_R(\gamma_-(0),\dot\gamma_-(0))\begin{pmatrix}1\\0\\kv_-^2\\-ku_-v_-\end{pmatrix}=v_-\begin{pmatrix}-\frac{(a^2+b^2)^{3/2}}{a^2}\left(\frac{\sin\beta\cos\beta}{b}+\frac{a}{a^2+b^2}\right)\cos\beta\\-\frac{(a^2+b^2)^{3/2}}{a^2}\left(\frac{\sin\beta\cos\beta}{b}+\frac{a}{a^2+b^2}\right)\sin\beta\\\frac{\sqrt{2gF(0)}}{b}v_-v_+\\-\frac{\sqrt{2gF(0)}}{b}v_-u_+\end{pmatrix}$$
$$=-\frac{v_-^2}{b}\begin{pmatrix}\frac{(a^2+b^2)^{3/2}}{a^2v_-}\left(\sin\beta\cos\beta+\frac{ab}{a^2+b^2}\right)\cos\beta\\\frac{(a^2+b^2)^{3/2}}{a^2v_-}\left(\sin\beta\cos\beta+\frac{ab}{a^2+b^2}\right)\sin\beta\\-\sqrt{2gF(0)}v_+\\\sqrt{2gF(0)}u_+\end{pmatrix}=-\frac{v_-^2}{b}\begin{pmatrix}\frac{(a^2+b^2)^{3/2}}{a^2}u_+\cos\beta\\\frac{(a^2+b^2)^{3/2}}{a^2}u_+\sin\beta\\-\sqrt{2gF(0)}v_+\\\sqrt{2gF(0)}u_+\end{pmatrix}$$

since

$$u_+v_-=\sin\beta\cos\beta+\frac{ab}{a^2+b^2}.$$

So

$$\mathcal{S}'''=R'\left(\gamma_-(0),\dot\gamma_-(0)\right)\mathcal{S}'=\operatorname{span}\left\{\begin{pmatrix}\frac{(a^2+b^2)^{3/2}\cos\beta}{a^2}u_+\\\frac{(a^2+b^2)^{3/2}\sin\beta}{a^2}u_+\\-\sqrt{2gF(0)}v_+\\\sqrt{2gF(0)}u_+\end{pmatrix}\right\}$$

Clearly $\dim[\mathcal{R}P_++\mathcal{S}''']=3$ if and only if

$$\operatorname{rank}\begin{pmatrix}u_+&v_+&\frac{(a^2+b^2)^{3/2}}{a^2}u_+\cos\beta\\v_+&-u_+&\frac{(a^2+b^2)^{3/2}}{a^2}u_+\sin\beta\\0&-\frac{\sqrt{2gF(0)}}{\|I_2-\left(\begin{smallmatrix}a\\b\end{smallmatrix}\right)\|}v_+&-\sqrt{2gF(0)}v_+\\0&\frac{\sqrt{2gF(0)}}{\|I_2-\left(\begin{smallmatrix}a\\b\end{smallmatrix}\right)\|}u_+&\sqrt{2gF(0)}u_+\end{pmatrix}=3 \tag{2.21}$$

Now:

$$\det\begin{pmatrix}u_+&v_+&\frac{(a^2+b^2)^{3/2}}{a^2}u_+\cos\beta\\v_+&-u_+&\frac{(a^2+b^2)^{3/2}}{a^2}u_+\sin\beta\\0&\frac{1}{\|I_2-\left(\begin{smallmatrix}a\\b\end{smallmatrix}\right)\|}&1\end{pmatrix}=-1-\frac{(a^2+b^2)^{3/2}}{a^2\|I_2-\left(\begin{smallmatrix}a\\b\end{smallmatrix}\right)\|}u_+(u_+\sin\beta-v_+\cos\beta)$$
$$=-1+\frac{(a^2+b^2)^{3/2}}{a^2\|I_2-\left(\begin{smallmatrix}a\\b\end{smallmatrix}\right)\|}u_+\frac{a}{\sqrt{a^2+b^2}}=-1+\frac{a^2+b^2}{a\|I_2-\left(\begin{smallmatrix}a\\b\end{smallmatrix}\right)\|}u_+$$
$$=-1-\frac{u_+}{u_-}=\frac{2a\sin\beta}{b\cos\beta-a\sin\beta}\neq 0.$$

So $\dim[\mathcal{R}P_++\mathcal{S}''']=3$ and condition (H3) of [2] is verified. Now, looking at the matrix in (2.21) we easily check that a unitary vector $\psi\in[\mathcal{R}P_++\mathcal{S}''']^\perp$ is given by:

$$\psi=\begin{pmatrix}0\\0\\u_+\\v_+\end{pmatrix}=-\frac{1}{\sqrt{2gF(0)}}\begin{pmatrix}0\\0\\\dot\gamma_+(0)\end{pmatrix} \tag{2.22}$$

3 Constructing the Melnikov function

Our purpose, in this paper, is to study the chaotic behaviour of the solutions of a small amplitude perturbation of equation (1.3) i.e. of

$$\begin{aligned} \ddot{x} &= \lambda f_x(x,y) + \varepsilon h_1(t,x,y,\dot{x},\dot{y},\varepsilon) \\ \ddot{y} &= \lambda f_y(x,y) + \varepsilon h_2(t,x,y,\dot{x},\dot{y},\varepsilon). \end{aligned} \tag{3.1}$$

We expect that if the perturbation is of sufficiently small amplitude and satisfies suitable recurrence conditions such as almost periodicity, the resulting equation exhibits chaotic behavior in the sense that a Smale-like horseshoe exists. To this end, according to [2, Theorem 4.2], we need to construct the Melnikov function associated to the perturbed equation, which, in turn, depends on the function

$$\psi(t) := \begin{cases} X_-^{-1}(t)^* P_-^* R_-^* DR(\gamma_-(0),\dot{\gamma}_-(0))^*\psi & \text{for } t \le 0 \\ \\ X_+^{-1}(t)^*(\mathbb{I} - P_+^*)\psi & \text{for } t > 0. \end{cases}$$

Here $X_\pm(t)$ are the fundamental matrices of the linear variational system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{y}_1 &= y_2 \\ \dot{x}_2 &= f_x(\gamma_\pm(t))\lambda'(\gamma_\pm(t),\dot{\gamma}_\pm(t)) \begin{pmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \end{pmatrix} + \lambda(\gamma_\pm(t),\dot{\gamma}_\pm(t))[f_{xx}(\gamma_\pm(t))x_1 + f_{xy}(\gamma_\pm(t))y_1] \\ \dot{y}_2 &= f_y(\gamma_\pm(t))\lambda'(\gamma_\pm(t),\dot{\gamma}_\pm(t)) \begin{pmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \end{pmatrix} + \lambda(\gamma_\pm(t),\dot{\gamma}_\pm(t))[f_{xy}(\gamma_\pm(t))x_1 + f_{yy}(\gamma_\pm(t))y_1] \end{aligned}$$

satisfying $X_\pm(0) = \mathbb{I}$, $P_\pm$ are the projections of the dichotomies of the linear variational system along $(\gamma_\pm(t),\dot{\gamma}_\pm(t))$ on $\mathbb{R}_\pm$, whose kernel and range have been described in (2.18), (2.19), and R_- is the projection onto $\mathcal{N}G'(\gamma_-(0),\dot{\gamma}_-(0))$ along $(\dot{\gamma}_-(0),\ddot{\gamma}_-(0))$.

First we simplify the expression of $\psi(t)$. From the expression of the Jacobian matrix $J_R(x,y,\mu u_-,\mu v_-)$ and (2.22) we see that

$$R'(\gamma_-(0),\dot{\gamma}_-(0))\psi = -\begin{pmatrix} 0 \\ 0 \\ u_- \\ v_- \end{pmatrix}.$$

Next, we already saw that

$$\nabla G(\gamma_-(0),\dot{\gamma}_-(0)) = \frac{\sqrt{a^2+b^2}}{v_-}\sin\beta \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

and

$$\begin{pmatrix} \dot{\gamma}_-(0) \\ \ddot{\gamma}_-(0) \end{pmatrix} = \sqrt{2gF(0)} \begin{pmatrix} u_- \\ v_- \\ 0 \\ 0 \end{pmatrix}.$$

So the matrix of R_- is given by:

$$\begin{pmatrix} 1 & -\frac{u_-}{v_-} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then

$$R_-^* R'(\gamma_-(0), \dot\gamma_-(0))\psi = -\begin{pmatrix} 0 \\ 0 \\ u_- \\ v_- \end{pmatrix}$$

and, finally:

$$P_-^* R_-^* DR(\gamma_-(0), \dot\gamma_-(0))\psi = -P_-^* \begin{pmatrix} 0 \\ 0 \\ u_- \\ v_- \end{pmatrix} = -\begin{pmatrix} 0 \\ 0 \\ u_- \\ v_- \end{pmatrix}$$

since we can take P_- to be the an orthogonal matrix and $\left(\begin{smallmatrix} 0 \\ 0 \\ u_- \\ v_- \end{smallmatrix}\right)$ is orthogonal to $\mathcal{N}P_-$ (see (2.18)). Similarly, we can take P_+ to be an orthogonal matrix and $\left(\begin{smallmatrix} 0 \\ 0 \\ u_+ \\ v_+ \end{smallmatrix}\right)$ is orthogonal to $\mathcal{R}P_+$. So:

$$\psi(t) = \begin{cases} -X_-^{-1}(t)^* \left(\begin{smallmatrix} 0 \\ 0 \\ u_- \\ v_- \end{smallmatrix}\right) & \text{if } t \le 0 \\ X_+^{-1}(t)^* \left(\begin{smallmatrix} 0 \\ 0 \\ u_+ \\ v_+ \end{smallmatrix}\right) & \text{if } t \ge 0 \end{cases} = -\frac{1}{\sqrt{2gF(0)}} \begin{cases} -X_-^{-1}(t)^* \left(\begin{smallmatrix} 0 \\ \dot\gamma_-(0) \end{smallmatrix}\right) & \text{if } t \le 0 \\ X_+^{-1}(t)^* \left(\begin{smallmatrix} 0 \\ \dot\gamma_+(0) \end{smallmatrix}\right) & \text{if } t \ge 0 \end{cases}$$

Note that $\psi_+(t) = X_+^{-1}(t)^* \left(\begin{smallmatrix} 0 \\ 0 \\ u_+ \\ v_+ \end{smallmatrix}\right)$ and $\psi_-(t) = X_-^{-1}(t)^* \left(\begin{smallmatrix} 0 \\ 0 \\ -u_- \\ -v_- \end{smallmatrix}\right)$ are the (bounded on $\mathbb{R}_\pm$) solutions of the adjoint variational system with the initial conditions $\left(\begin{smallmatrix} 0 \\ 0 \\ u_+ \\ v_+ \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 \\ 0 \\ -u_- \\ -v_- \end{smallmatrix}\right)$ respectively.

We now show how the existence of first integrals can be used to obtain $\psi_\pm(t)$ without the need of computing $X_\pm(t)$. We will need the following

Theorem 3.1. *Let $z_0(t)$ be a solution on an interval $I_0 \subseteq \mathbb{R}$ of the ODE $\dot z = g(z)$, with $z \in \mathbb{R}^N$ and g of class C^1. Suppose that the equation has a smooth first integral J of class C^2, i.e., $\langle g(z), \nabla J(z)\rangle = 0$ for any $z \in \mathbb{R}^N$. Then $w(t) = \nabla J(z_0(t))$ is a solution on I_0 of the adjoint system $w = -Dg(z_0(t))^* w$ along $z_0(t)$.*

Proof. Let H_J be the Hessian of J. Differentiating $\langle g(z), \nabla J(z)\rangle = 0$ for any $z \in \mathbb{R}^N$ we get

$$\langle Dg(z)v, \nabla J(z)\rangle + \langle g(z), H_J(z)v\rangle = 0$$

for any $z, v \in \mathbb{R}^N$. This is equivalent to

$$Dg(z)^* \nabla J(z) + H_J(z)g(z) = 0$$

for any $z \in \mathbb{R}^N$. Then we derive

$$\dot w(t) = H_J(z_0(t))\dot z_0(t) = H_J(z_0(t))g(z_0(t)) = -Dg(z_0(t))^* \nabla J(z_0(t)) = -Dg(z_0(t))^* w(t).$$

The proof is finished. ∎

To apply Theorem 3.1 to our case we observe that, as it is well known, the motion of our particle for (1.1) has the energy

$$\frac{\dot x^2}{2} + \frac{\dot y^2}{2} + \frac{\dot z^2}{2} + gz$$

consisting of the kinetic and potential parts. Hence (1.3) has the first integral

$$H = \frac{\dot x^2}{2} + \frac{\dot y^2}{2} + \frac{(\dot x f_x(x,y) + \dot y f_y(x,y))^2}{2} + gf(x,y)$$

Corresponding to H we have the Lagrangian:

$$L(x,y,\dot{x},\dot{y}) = \frac{\dot{x}^2}{2} + \frac{\dot{y}^2}{2} + \frac{(\dot{x}f_x(x,y)+\dot{y}f_y(x,y))^2}{2} - gf(x,y)$$

and the Euler-Lagrangian equation is (1.3). We plug (1.4) into the above formulas to get:

$$\begin{aligned} H &= \frac{1}{2}\left(X^2+Y^2\right) + 2\left\langle \begin{pmatrix} X \\ Y \end{pmatrix}, \begin{pmatrix} x-a \\ y-b \end{pmatrix} \right\rangle^2 F'[(x-a)^2+(y-b)^2]^2 + gF[(x-a)^2+(y-b)^2] \\ L &= \frac{1}{2}\left(X^2+Y^2\right) + 2\left\langle \begin{pmatrix} X \\ Y \end{pmatrix}, \begin{pmatrix} x-a \\ y-b \end{pmatrix} \right\rangle^2 F'[(x-a)^2+(y-b)^2]^2 - gF[(x-a)^2+(y-b)^2] \end{aligned}$$

with $X=\dot{x}$ and $Y=\dot{y}$. We take $H(x,y,X,Y)$ as J and derive

$$\begin{aligned} H'(x,y,X,Y) = \Bigg(& 2F'((a-x)^2+(b-y)^2)\Big(2X(-aX+xX-bY+yY)F'((a-x)^2+(b-y)^2) \\ & +(a-x)\left(-g-4(aX-xX+bY-yY)^2F''((a-x)^2+(b-y)^2)\right)\Big), \\ & 2F'((a-x)^2+(b-y)^2)\Big(2Y(-aX+xX-bY+yY)F'((a-x)^2+(b-y)^2) \\ & +(b-y)\left(-g-4(aX-xX+bY-yY)^2F''((a-x)^2+(b-y)^2)\right)\Big), \\ & X+4(a-x)(aX-xX+bY-yY)F'((a-x)^2+(b-y)^2)^2, \\ & Y+4(b-y)(aX-xX+bY-yY)F'((a-x)^2+(b-y)^2)^2\Bigg). \end{aligned} \tag{3.2}$$

Inserting $(x,y,X,Y)=(\gamma_+(t),\dot{\gamma}_+(t))$ into (3.2) and using (2.3), (2.6), (2.20), and the definition of u_+,v_+ we derive

$$\begin{aligned} \nabla H(\gamma_+(t),\dot{\gamma}_+(t)) &= \begin{pmatrix} 2u_+\rho_+F'(\rho_+^2)\left(g+2\dot{\rho}_+^2\left(F'(\rho_+^2)+2\rho_+^2F''(\rho_+^2)\right)\right) \\ 2v_+\rho_+F'(\rho_+^2)\left(g+2\dot{\rho}_+^2\left(F'(\rho_+^2)+2\rho_+^2F''(\rho_+^2)\right)\right) \\ u_+\left(1+4\rho_+^2F'(\rho_+^2)^2\right)\dot{\rho}_+ \\ v_+\left(1+4\rho_+^2F'(\rho_+^2)^2\right)\dot{\rho}_+ \end{pmatrix} \\ &= \left(1+4\rho_+^2F'(\rho_+^2)^2\right)\left(-u_+\ddot{\rho}_+,-v_+\ddot{\rho}_+,u_+\dot{\rho}_+,v_+\dot{\rho}_+\right)^* \end{aligned}$$

where $\rho_+=\rho_+(t):=\rho_0(t^+_{(a,b)}+t)$. Similarly, with $(x,y,X,Y)=(\gamma_-(t),\dot{\gamma}_-(t))$:

$$\nabla H(\gamma_-(t),\dot{\gamma}_-(t)) = \left(1+4\rho_-^2F'(\rho_-^2)^2\right)\left(-u_-\ddot{\rho}_-,-v_-\ddot{\rho}_-,u_-\dot{\rho}_-,v_-\dot{\rho}_-\right)^*,$$

where $\rho_-=\rho_-(t):=\rho_0(t^-_{(a,b)}-t)$. From (2.20) we also get

$$\gamma_\pm(t) = \left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right) + \rho_\pm(t)\left(\begin{smallmatrix} u_\pm \\ v_\pm \end{smallmatrix}\right).$$

Now, Theorem 3.1 implies that $\nabla H(\gamma_-(t),\dot{\gamma}_-(t))$ and $\nabla H(\gamma_-(t),\dot{\gamma}_-(t))$ are (bounded) solutions of the adjoint equation on the intervals $(-\infty,0]$ and $[0,\infty)$ respectively. Moreover (see also (2.15))

$$\nabla H(\gamma_\pm(0),\dot{\gamma}_\pm(0)) = \left(1+4\rho_\pm(0)^2F'(\rho_\pm(0)^2)^2\right)\left(0,0,u_\pm\dot{\rho}_\pm(0),v_+\dot{\rho}_\pm(0)\right)^* = -\sqrt{2gF(0)}\psi(0^\pm).$$

and then we get:

$$\psi(t) = -\frac{1}{\sqrt{2gF(0)}}\nabla H(\gamma(t),\dot{\gamma}(t)), \quad t\neq 0.$$

We can further simplify the expression $\psi(t)$ observing that, from (1.4) it follows:

$$\begin{aligned}\nabla H(\gamma_\pm(t),\dot\gamma_\pm(t)) &= -\left(1+4\rho_\pm^2(t)F'(\rho_\pm^2(t))^2\right)\begin{pmatrix}\ddot\rho_\pm(t)\left(\begin{smallmatrix}u_\pm\\ v_\pm\end{smallmatrix}\right)\\ -\dot\rho_\pm(t)\left(\begin{smallmatrix}u_\pm\\ v_\pm\end{smallmatrix}\right)\end{pmatrix}\\ &= [1+\|\nabla f(\gamma_\pm(t))\|^2]\begin{pmatrix}\ddot\rho_\pm(t)\left(\begin{smallmatrix}u_\pm\\ v_\pm\end{smallmatrix}\right)\\ -\dot\rho_\pm(t)\left(\begin{smallmatrix}u_\pm\\ v_\pm\end{smallmatrix}\right)\end{pmatrix}.\end{aligned}$$

So:

$$\psi(t)=\frac{1+\|\nabla f(\gamma_\pm(t))\|^2}{\sqrt{2gF(0)}}\begin{pmatrix}-\ddot\rho_\pm(t)\left(\begin{smallmatrix}u_\pm\\ v_\pm\end{smallmatrix}\right)\\ \dot\rho_\pm(t)\left(\begin{smallmatrix}u_\pm\\ v_\pm\end{smallmatrix}\right)\end{pmatrix}=\frac{1+\|\nabla f(\gamma_\pm(t))\|^2}{\sqrt{2gF(0)}}\begin{pmatrix}-\ddot\gamma_\pm(t)\\ \dot\gamma_\pm(t)\end{pmatrix},\quad t\neq 0. \tag{3.3}$$

Then, according to [2, Theorem 4.2] the Melnikov function characterizing chaotic behavior of the solutions of (3.1) is:

$$\begin{aligned}\mathcal{M}(\alpha) &= \int_{-\infty}^{\infty}\psi^*(t)\begin{pmatrix}0\\0\\ h_1(t+\alpha,\gamma(t),\dot\gamma(t),0)\\ h_2(t+\alpha,\gamma(t),\dot\gamma(t),0)\end{pmatrix}dt\\ &= \frac{1}{\sqrt{2gF(0)}}\left[\int_{-\infty}^{\infty}[1+\|\nabla f(\gamma(t))\|^2]\langle\dot\gamma(t),\left(\begin{smallmatrix}h_1(t+\alpha,\gamma(t),\dot\gamma(t),0)\\ h_2(t+\alpha,\gamma(t),\dot\gamma(t),0)\end{smallmatrix}\right)\rangle dt\right]\end{aligned}\tag{3.4}$$

where we use the notation of (2.14).

We conclude this Section with a remark. Equation (1.3) has another first integral, independent of H that can be constructed as follows. Consider the one parameter family of rotations in $\mathbb{R}^2$ given by

$$A(s)\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}a\\b\end{pmatrix}+B(s)\begin{pmatrix}x-a\\y-b\end{pmatrix},\quad B(s)=\begin{pmatrix}\cos s&\sin s\\-\sin s&\cos s\end{pmatrix}.$$

Then H and L are invariant under $A(s)$ in the sense that

$$H(A(s)(x,y),B(s)(X,Y))=H(x,y,X,Y),\quad L(A(s)(x,y),B(s)(X,Y))=L(x,y,X,Y).$$

Note $D_{(x,y)}\left[A(s)\left(\begin{smallmatrix}x\\y\end{smallmatrix}\right)\right]=B(s)$. From Noether theorem [1], we know that

$$\begin{aligned}I(x,y,X,Y) &= D_{(X,Y)}L(x,y,X,Y)A'(0)\begin{pmatrix}x\\y\end{pmatrix}\\ &= \begin{pmatrix}X+4(a-x)(aX-xX+bY-yY)F'((a-x)^2+(b-y)^2)^2\\ Y+4(b-y)(aX-xX+bY-yY)F'((a-x)^2+(b-y)^2)^2\end{pmatrix}\begin{pmatrix}y-b\\a-x\end{pmatrix}\\ &= -bX+Xy+(a-x)Y\end{aligned}$$

is a first integral of (1.3). Note that, with $x=x_1$, $y=y_1$, $X=x_2$ and $Y=y_2$, $I(x,y,X,Y)$ coincides with (2.5), i.e. I comes from the radial symmetry, as it could be expected.

Applying again Theorem 3.1 to (2.17) with the first integral $I(x,y,X,Y)$ and using (2.20), we obtain the following bounded solutions of the adjoint variational system on $(-\infty,0]$ and $[0,\infty)$ respectively:

$$\nabla I(\gamma_-(t),\dot\gamma_-(t))=\begin{pmatrix}\dot\rho_0(t_{(a,b)}^- - t)\left(\begin{smallmatrix}v_-\\ -u_-\end{smallmatrix}\right)\\ \rho_0(t_{(a,b)}^- - t)\left(\begin{smallmatrix}v_-\\ -u_-\end{smallmatrix}\right)\end{pmatrix},\qquad \nabla I(\gamma_+(t),\dot\gamma_+(t))=\begin{pmatrix}-\dot\rho_0(t_{(a,b)}^+ + t)\left(\begin{smallmatrix}v_+\\ -u_+\end{smallmatrix}\right)\\ \rho_0(t_{(a,b)}^+ + t)\left(\begin{smallmatrix}v_+\\ -u_+\end{smallmatrix}\right)\end{pmatrix}.$$

but the function

$$w(t):=\begin{cases}\nabla I(\gamma_-(t),\dot\gamma_-(t)) & \text{for } t\le 0\\ \nabla I(\gamma_+(t),\dot\gamma_+(t)) & \text{for } t>0\end{cases}$$

is not a bounded solution of the variational equation (see [2, eq.(45)]) since it does not satisfy the impact condition (see the 2nd equation of [2, eq.(45)])

$$R_-^*[R'(\gamma_-(0),\dot\gamma_-(0))^*w(0^+)-w(0^-)]=0. \tag{3.5}$$

Indeed:

$$R_-^*[R'(\gamma_-(0),\dot\gamma_-(0))^* w(0^+) - w(0^-)]$$
$$= (0,0,2\sin\beta(a\cos\beta + b\sin\beta), 2\sin\beta(b\cos\beta - a\sin\beta))^* \neq 0.$$

On the other hand, it is true that $R_-^*[R'(\gamma_-(0),\dot\gamma_-(0))^*\psi(0^+) - \psi(0^-)] = 0$ as we already know. Thus to apply Theorem 3.1 the knowledge of a first integral it is not enough to obtain a solution $w(t)$ of the adjoint variational equation as given in the 1th and 3rd equations of [2, eq.(45)], but one also has to check weather $w(t)$ satisfies the impact condition (3.5) presented by the 2nd equation of [2, eq.(45)]).

4 Chaotic behavior

In this Section we perturb equation (1.3), or equivalently (2.17), and construct the corresponding Melnikov function associated to the chaotic behavior of the perturbed system. We construct such a perturbation by allowing the boundary of Ω to oscillate around the equilibrium. However to fit into the framework of this paper we need that the perturbation does not act when the particle runs from the first hitting line ($y = 0$) to the second ($y = x\tan\beta$). We may obtain such a situation endowing the line $y = 0$ with a switcher interrupting the boundary movement when the particle hits it and another switch restoring the boundary movement immediately before the time when the particle hits the line $y = x\tan\beta$. Another kind of perturbations fitting into this framework may be obtained by taking a steel particle and letting the gravity acceleration g vary slowly by the effect of a electromagnetic field. Again we need two switchers near the impact boundary: the first (near the line $y = 0$) stopping the electromagnetic field and the second restoring it immediately before the particle hits the line $y = x\tan\beta$. If these conditions are satisfied we may neglect the part of the trajectory from $y = 0$ to $y = x\tan\beta$ and assume that the impact manifold varies with time as follows:

$$\mathcal{I} = \{(y_1 - \varepsilon p(t))(y_1 - x_1\tan\beta - \varepsilon p(t)) = 0\}.$$

where $p(t)$ is a periodic (or almost periodic) C^4-function.

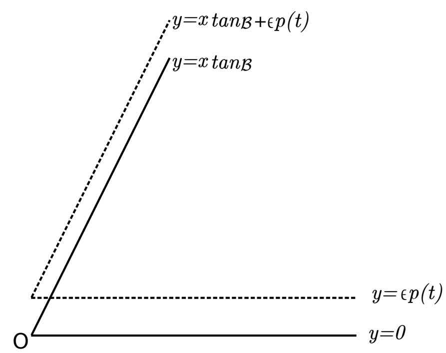

Figure 4: The billiard with a moving boundary

Remark 4.1. Suppose a solution hits the line $y = 0$ at a point x_0 near the homoclinic solution with speed $(\dot x_0, \dot y_0)$. Then it is reflected to the solution starting from $(x_0, 0)$ with speed $(\dot x_0, -\dot y_0)$. Since this solution satisfies $\ddot x = \ddot y = 0$ we have:

$$x(t) = x_0 + t\dot x_0, \qquad y(t) = -t\dot y_0.$$

So the reflected solution hits the line $y = x\tan\beta$ at the time t such that

$$-t\dot y_0 = (x_0 + t\dot x_0)\tan\beta \Leftrightarrow t = -\frac{x_0}{\dot x_0 + \frac{\dot y_0}{\tan\beta}}.$$

So, in our framework, we essentially assume that the perturbation (or boundary of the billiard) stops for a time duration given by $-\frac{x_0}{\dot{x}_0+\frac{\dot{y}_0}{\tan\beta}}$ before starting again, where x_0 is the point of the x axis hit by the ball $\dot{x}_0, \dot{y}_0$ is the speed of the ball at the hitting time.

Changing $y_1(t)$ and $y_2(t)$ with $y_1 - \varepsilon p(t)$ and $y_2 - \varepsilon \dot{p}(t)$ respectively we obtain the system, instead of (2.17):

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{y}_1 &= y_2 \\
\dot{x}_2 &= \lambda(x_1, y_1 + \varepsilon p(t), x_2, y_2 + \varepsilon \dot{p}(t)) f_x(x_1, y_1 + \varepsilon p(t)) \\
\dot{y}_2 &= \lambda(x_1, y_1 + \varepsilon p(t), x_2, y_2 + \varepsilon \dot{p}(t)) f_y(x_1, y_1 + \varepsilon p(t)) - \varepsilon \ddot{p}(t)
\end{aligned} \tag{4.1}$$

that we write as:

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{y}_1 &= y_2 \\
\dot{x}_2 &= \lambda(x_1, y_1, x_2, y_2) f_x(x_1, y_1) + \varepsilon h_1(t, x_1, y_1, x_2, y_2, \varepsilon) \\
\dot{y}_2 &= \lambda(x_1, y_1, x_2, y_2) f_y(x_1, y_1) + \varepsilon h_2(t, x_1, y_1, x_2, y_2, \varepsilon) - \varepsilon \ddot{p}(t)
\end{aligned} \tag{4.2}$$

This change of variables has the effect that the domain Ω does not change and it is as in the previous Section: $\Omega = \{(x_1, x_2, y_1, y_2) \mid y_1(y_1 - x_1 \tan\beta) > 0\}$. So the perturbed impact equation is (4.2) together with the same impact conditions as the unperturbed system:

$$y_1(t^{*-}) = 0 \Rightarrow (x_1(t^{*+}), y_1(t^{*+}), x_2(t^{*+}), y_2(t^{*+})) = R(x_1(t^{*-}), 0, x_2(t^{*-}), y_2(t^{*-})).$$

Let $z = (x_1, y_1, x_2, y_2)$, $\zeta(t) = (0, p(t), 0, \dot{p}(t))$ and

$$F(z) = \begin{pmatrix} x_2 \\ y_2 \\ \lambda(z) f_x(x_1, y_1) \\ \lambda(z) f_y(x_1, y_1) \end{pmatrix}$$

Taking $z + \varepsilon\zeta(t) = (x_1, y_1 + \varepsilon p(t), x_2, y_2 + \varepsilon \dot{p}(t))$ we obtain

$$F(z + \varepsilon\zeta(t)) - \varepsilon\dot{\zeta}(t) = \begin{pmatrix} x_2 \\ y_2 + \varepsilon \dot{p}(t) \\ \lambda(z + \varepsilon\zeta(t)) f_x(x_1, y_1 + \varepsilon p(t)) \\ \lambda(z + \varepsilon\zeta(t)) f_y(x_1, y_1 + \varepsilon p(t)) \end{pmatrix} - \varepsilon \begin{pmatrix} 0 \\ \dot{p}(t) \\ 0 \\ \ddot{p}(t) \end{pmatrix}$$

So that our perturbed system reads

$$\dot{z} = F(z + \varepsilon\zeta(t)) - \varepsilon\dot{\zeta}(t) \tag{4.3}$$

that we write as

$$\dot{z} = F(z) + \varepsilon \begin{pmatrix} 0 \\ 0 \\ h_1(t, z, \varepsilon) \\ h_2(t, z, \varepsilon) \end{pmatrix}$$

Note

$$\begin{pmatrix} 0 \\ 0 \\ h_1(t, \gamma_\pm(t), \dot{\gamma}_\pm(t), 0) \\ h_2(t, \gamma_\pm(t), \dot{\gamma}_\pm(t), 0) \end{pmatrix} = F'(\gamma_\pm(t), \dot{\gamma}_\pm(t))\zeta(t) - \dot{\zeta}(t).$$

The Melnikov function associated to the perturbed problem (4.3) is then (see (3.4))

$$
\begin{aligned}
\mathcal{M}(\alpha) &:= \int_0^\infty \psi_+^*(t)\left(F'(\gamma_+(t),\dot\gamma_+(t))\zeta(t+\alpha)-\dot\zeta(t+\alpha)\right)dt \\
&+\int_{-\infty}^0 \psi_-^*(t)\left(F'(\gamma_-(t),\dot\gamma_-(t))\zeta(t+\alpha)-\dot\zeta(t+\alpha)\right)dt = \\
&\int_0^\infty \psi_+^*(t)F'(\gamma_+(t),\dot\gamma_+(t))\zeta(t+\alpha)dt-\int_0^\infty \psi_+^*(t)\dot\zeta(t+\alpha)dt \\
&+\int_{-\infty}^0 \psi_-^*(t)F'(\gamma_-(t),\dot\gamma_-(t))\zeta(t+\alpha)dt-\int_{-\infty}^0 \psi_-^*(t)\dot\zeta(t+\alpha)dt = \\
&\int_0^\infty \left(F'(\gamma_+(t),\dot\gamma_+(t))\psi_+(t)\right)^*\zeta(t+\alpha)dt-\int_0^\infty \psi_+^*(t)\dot\zeta(t+\alpha)dt \\
&+\int_{-\infty}^0 \left(F'(\gamma_-(t),\dot\gamma_-(t))\psi_-(t)\right)^*\zeta(t+\alpha)dt-\int_{-\infty}^0 \psi_-^*(t)\dot\zeta(t+\alpha)dt = \\
-\int_0^\infty \dot\psi_+^*(t)\zeta(t+\alpha)dt &-\int_0^\infty \psi_+^*(t)\dot\zeta(t+\alpha)dt-\int_{-\infty}^0 \dot\psi_-^*(t)\zeta(t+\alpha)dt-\int_{-\infty}^0 \psi_-^*(t)\dot\zeta(t+\alpha)dt = \\
&=-\int_0^\infty \frac{d}{dt}\left[\psi_+^*(t)\zeta(t+\alpha)\right]dt-\int_{-\infty}^0 \frac{d}{dt}\left[\psi_-^*(t)\zeta(t+\alpha)\right]dt \\
&=\left(\psi_+(0)-\psi_-(0)\right)^*\zeta(\alpha)=(v_+-v_-)\dot p(\alpha)=\frac{2a\cos\beta\dot p(\alpha)}{\sqrt{a^2+b^2}}.
\end{aligned}
$$

As a consequence $\mathcal{M}(\alpha)$ has a simple zero at some α if and only if $p(\alpha)$ has a non degenerate max or min. So using [2, Theorem 4.2], we conclude with the following

Theorem 4.2. *Assume that $F(\rho)$ is a C^5 function whose support is contained in the interval $[0, r_0^2]$ such that $\overline{B((a,b),r_0)} \subset \overset{\circ}{\Omega}$ and such that $F'(0) < 0$ and $p(t)$ is an almost periodic C^4 function with a non degenerate* max *or* min*. Then there exists $\varepsilon_0 > 0$ such that for $|\varepsilon| < \varepsilon_0$ equation* (4.1) *behaves chaotically in a suitable neighborhood of the impact homoclinic orbit $(\gamma(t), \dot\gamma(t))$.*

As a second example we consider the case of the periodically perturbed gravity g. So we assume g is changed with $g + \varepsilon p(t)$ and $p(t)$ is a C^2, $T-$periodic, function. Then, emphasizing dependence on g the perturbed system is:

$$
\begin{aligned}
\dot x_1 &= x_2 \\
\dot y_1 &= y_2 \\
\dot x_2 &= \lambda(x_1, y_1, x_2, y_2, g+\varepsilon p(t)) f_x(x_1, y_1) \\
\dot y_2 &= \lambda(x_1, y_1, x_2, y_2, g+\varepsilon p(t)) f_y(x_1, y_1).
\end{aligned}
\tag{4.4}
$$

So the perturbation is given by:

$$
\varepsilon^{-1}[\lambda(x_1,y_1,x_2,y_2,g+\varepsilon p(t+\alpha))-\lambda(x_1,y_1,x_2,y_2,g)]\begin{pmatrix}0\\0\\f_x(x_1,y_1)\\f_y(x_1,y_1)\end{pmatrix}
$$

and its limit as $\varepsilon \to 0$ evaluated at $(\gamma(t), \dot\gamma(t))$ is:

$$
\frac{\partial\lambda}{\partial g}(\gamma_\pm(t),\dot\gamma_\pm(t),g)p(t+\alpha)\begin{pmatrix}0\\0\\f_x(\gamma_\pm(t))\\f_y(\gamma_\pm(t))\end{pmatrix} = -\frac{1}{1+\|\nabla f(\gamma_\pm(t))\|^2}\begin{pmatrix}0\\\nabla f(\gamma_\pm(t))\end{pmatrix}p(t+\alpha)
$$

Now:

$$
\left\langle \begin{pmatrix}-\ddot\gamma_\pm(t)\\\dot\gamma_\pm(t)\end{pmatrix}, \begin{pmatrix}0\\\nabla f(\gamma_\pm(t))\end{pmatrix}\right\rangle = \langle\dot\gamma_\pm(t),\nabla f(\gamma_\pm(t))\rangle = \frac{d}{dt}f(\gamma_\pm(t)).
$$

So:

$$\int_0^\infty \psi^*(t)\frac{-1}{1+\|\nabla f(\gamma_+(t))\|^2}\begin{pmatrix}0\\ \nabla f(\gamma_+(t))\end{pmatrix}p(t+\alpha)dt = \frac{-1}{\sqrt{2gF(0)}}\int_0^\infty p(t+\alpha)\frac{d}{dt}f(\gamma_+(t))dt$$
$$= \frac{1}{\sqrt{2gF(0)}}\left(p(\alpha)[f(\gamma_+(0))-f(a,b)]+\int_0^\infty[f(\gamma_+(t))-f(a,b)]\dot p(t+\alpha)dt\right)$$

and similarly:

$$\int_{-\infty}^0 \psi^*(t)\frac{-1}{1+\|\nabla f(\gamma_-(t))\|^2}\begin{pmatrix}0\\ \nabla f(\gamma_-(t))\end{pmatrix}p(t+\alpha)dt$$
$$= \frac{1}{\sqrt{2gF(0)}}\left(-p(\alpha)[f(\gamma_-(0))-f(a,b)]+\int_{-\infty}^0[f(\gamma_-(t))-f(a,b)]\dot p(t+\alpha)dt\right).$$

Note $f(a,b)=F(0)$. Since the support of $f(x,y)$ is contained in the ball $(x-a)^2+(y-b)^2\le r_0^2$ and $\gamma_\pm(0)$ do not belong to this ball we get:

$$\mathcal{M}(\alpha)=\frac{1}{\sqrt{2gF(0)}}\int_{-\infty}^\infty(f(\gamma(t))-f(a,b))\dot p(t+\alpha)dt=\frac{1}{\sqrt{2gF(0)}}\lim_{\mathbb{N}\ni k\to\infty}\int_{-kT}^{kT}f(\gamma(t))\dot p(t+\alpha)dt.$$

Then we see that

$$\int_{-T}^T\mathcal{M}(\alpha)d\alpha=0$$

so that if $\mathcal{M}(\alpha)\not\equiv 0$ the equation $\mathcal{M}(\alpha)=0$ must have a solution that generically is simple. As a consequence, using again [2, Theorem 4.2], we get the following result.

Theorem 4.3. *Assume that $F(\rho)$ is a C^5 function whose support is contained in the interval $[0,r_0^2]$ such that $\overline{B((a,b),r_0)}\subset\overset{\circ}{\Omega}$ and such that $F'(0)<0$. Then, given any $p(t)$ in a open dense subset of the space of C^2, $T-$periodic functions, there exists $\varepsilon_0>0$ such that for $|\varepsilon|<\varepsilon_0$ equation (4.4) behaves chaotically in a suitable neighborhood of the impact homoclinic orbit $(\gamma(t),\dot\gamma(t))$.*

We continue with Example 1.1. To allow more generality we take: $a=\cos\theta$, $b=\sin\theta$ and $\beta=2\theta$ with $\theta\in\left(0,\frac{\pi}{4}\right)$. Then condition (2.13) reads $r_0<\min\left\{2\sin^2\theta,\sin\theta\right\}$, so that we take $r_0=\sin^2\theta$. Moreover, we derive $t_{a,b}^\pm=t_{a,b}=\frac{r_0-\tan\theta}{\sqrt{2F(0)g}}<0$ and thus $\rho_+(t)=\rho_-(-t)$. Recall that

$$\rho_+(t)=\rho_0(t_{(a,b)}^+ + t)\quad\text{and}\quad \rho_-(t)=\rho_0(t_{(a,b)}^- - t).$$

Hence we arrive at

$$\mathcal{M}(\alpha)=\frac{1}{\sqrt{2gF(0)}}\left(\int_{-\infty}^0(F(\rho_-(t))-F(0))\dot p(t+\alpha)dt+\int_0^\infty(F(\rho_+(t))-F(0))\dot p(t+\alpha)dt\right)$$
$$=\frac{1}{\sqrt{2gF(0)}}\int_0^\infty\left(F(\rho_+(t)^2)-F(0)\right)(\dot p(t+\alpha)+\dot p(-t+\alpha))\,dt.$$

So if $p(t)$ is even then $\mathcal{M}(0)=0$ and

$$\mathcal{M}'(0)=\frac{\sqrt2}{\sqrt{gF(0)}}\int_0^\infty\left(F(\rho_+(t)^2)-F(0)\right)\ddot p(t)dt$$
$$=\frac{\sqrt2}{\sqrt{gF(0)}}\Big(-F(0)\dot p(-t_{a,b})+\int_{-t_{a,b}}^\infty\left(F(\rho_+(t)^2)-F(0)\right)\ddot p(t)dt\Big)$$
$$=\frac{\sqrt2}{\sqrt{gF(0)}}\Big(-F(0)\dot p(-t_{a,b})+\int_0^\infty\left(F(\rho_0(t)^2)-F(0)\right)\ddot p(t-t_{a,b})dt\Big).$$

Note $\rho_0(t)$ is determined by

$$\dot\rho_0 = -G(\rho_0^2)\rho_0, \quad \rho_0(0) = r_0$$

with

$$G(r) := \begin{cases} \sqrt{\frac{2g(F(0)-F(r))}{r(1+4rF'(r)^2)}} & \text{for } r > 0, \\ \sqrt{-2gF'(0)} & \text{for } r = 0. \end{cases}$$

and $0 < \rho_0(t) < r_0$ for $t > 0$. Set

$$\bar G := \max_{r\in[0,r_0^2]} G(r), \quad \bar g := \min_{r\in[0,r_0^2]} G(r),$$
$$\bar K := \sup_{r\in(0,r_0^2]} \frac{F(0)-F(r)}{r} > 0, \quad \bar k := \inf_{r\in(0,r_0^2]} \frac{F(0)-F(r)}{r} > 0$$

(recall that $F'(0) < 0$). Then clearly $r_0\,\mathrm{e}^{-\bar G t} \le \rho_0(t) \le r_0\,\mathrm{e}^{-\bar g t}$ for all $t \ge 0$. Now assume

$$\ddot p(-t_{a,b}) > 0, \tag{4.5}$$

and take the smallest $t_0 > 0$ such that $\ddot p(t - t_{a,b}) > 0$ for any $t \in [0, t_0)$ and $\ddot p(t_0 - t_{a,b}) = 0$. Note such $t_0 > 0$ exists since $\int_0^T \ddot p(t - t_{a,b})dt = 0$. Next, since $\bar k r \le F(0) - F(r) \le \bar K r$, we get $\bar k \rho_0(t)^2 \le F(0) - F(\rho_0(t)^2) \le \bar K \rho_0(t)^2$ and then

$$\begin{aligned}&\int_0^\infty (F(0) - F(\rho_0(t)^2))\ddot p(t - t_{a,b})dt \ge \bar k \int_0^{t_0} \rho_0(t)^2 \ddot p(t - t_{a,b})dt - \bar K \int_{t_0}^\infty \rho_0(t)^2 |\ddot p(t - t_{a,b})|dt \\ &\ge r_0^2 \bar k \int_0^{t_0} \mathrm{e}^{-2\bar G t}\, \ddot p(t - t_{a,b})dt - r_0^2 \bar K \int_{t_0}^\infty \mathrm{e}^{-2\bar g t}\, |\ddot p(t - t_{a,b})|dt\end{aligned}$$

Then it holds

$$\mathcal{M}'(0) \le -\frac{\sqrt2}{\sqrt{gF(0)}} \left(F(0)\dot p(-t_{a,b}) + r_0^2 \bar k \int_0^{t_0} \mathrm{e}^{-2\bar G t}\, \ddot p(t - t_{a,b})dt - r_0^2 \bar K \int_{t_0}^\infty \mathrm{e}^{-2\bar g t}\, |\ddot p(t - t_{a,b})|dt \right).$$

Hence, if in addition it holds

$$F(0)\dot p(-t_{a,b}) + r_0^2 \bar k \int_0^{t_0} \mathrm{e}^{-2\bar G t}\, \ddot p(t - t_{a,b})dt - r_0^2 \bar K \int_{t_0}^\infty \mathrm{e}^{-2\bar g t}\, |\ddot p(t - t_{a,b})|dt > 0, \tag{4.6}$$

we can apply Theorem 4.3 and conclude that equation(4.4) behaves chaotically. For Example 1.1, we compute

$$\bar g \doteq 4.2335, \quad \bar G \doteq 21.5527, \quad \bar k = 16, \quad \bar K = 96.$$

Recall $t_{a,b} \doteq -0.0739158$. Finally we take $p(t) = -\cos t$. Then $\dot p(t) = \sin t$ and $\ddot p(t) = \cos t$. Next, $\ddot p(-t_{a,b}) \doteq \cos 0.0739158 \doteq 0.997269 > 0$, so (4.5) holds. Clearly $t_0 = \frac{\pi}{2} + t_{a,b} \doteq 1.49688$. Then we can check that

$$F(0)\dot p(-t_{a,b}) + r_0^2 \bar k \int_0^{t_0} \mathrm{e}^{-2\bar G t}\, \ddot p(t - t_{a,b})dt - r_0^2 \bar K \int_{t_0}^\infty \mathrm{e}^{-2\bar g t}\, |\ddot p(t - t_{a,b})|dt \doteq 0.0969298 > 0$$

so assumption (4.6) is verified as well.

5 Using symmetries to obtain the Melnikov function

In this Section we suggest a direct way to obtain the function $\psi(t)$ which is not based of the existence of first integrals. This approach is based on a careful analysis of the variational system of the first order equations associated to equation (1.3) along the solution $(\gamma_\pm(t), \dot\gamma_\pm(t))$. It is proved that this variational equation satisfies symmetry conditions that allow us to reduce its order from the 4th to the 2nd. Obviously this methods applies any times the 4th order variational equation has the form (5.1) and the coefficient matrix satisfies suitable symmetry conditions (see equations (5.2) and (5.3)).

From (2.3) we know that

$$\lambda(x_1,y_1,x_2,y_2) = \lambda(\rho^2,\eta^2,\theta^2) := -\frac{g + 2F'\left(\rho^2\right)\left(\rho^2\theta^2+\eta^2\right) + 4\rho^2\eta^2F''\left(\rho^2\right)}{1+4\rho^2F'\left(\rho^2\right)^2}$$

with $\rho^2 = (x_1-a)^2 + (y_1-b)^2$, $\eta^2 = x_2^2 + y_2^2$ and $\rho^2\theta = y_2(x_1-a) - x_2(y_1-b)$. Then:

$$\rho\nabla\rho = \begin{pmatrix} x_1-a \\ y_1-b \\ 0 \\ 0 \end{pmatrix}, \qquad \eta\nabla\eta = \begin{pmatrix} 0 \\ 0 \\ x_2 \\ y_2 \end{pmatrix}, \qquad 2\theta\rho\nabla\rho + \rho^2\nabla\theta = \begin{pmatrix} y_2 \\ -x_2 \\ -(y_1-b) \\ x_1-a \end{pmatrix}$$

(note that the derivatives are taken with respect to all variables (x_1,y_1,x_2,y_2), we omitted the argument (x_1,y_1,x_2,y_2) for simplicity). So

$$\rho^2\nabla\theta = \begin{pmatrix} y_2 \\ -x_2 \\ -(y_1-b) \\ x_1-a \end{pmatrix} - 2\theta\begin{pmatrix} x_1-a \\ y_1-b \\ 0 \\ 0 \end{pmatrix}.$$

Now:

$$\nabla\lambda = 2D_1\lambda(\rho^2,\eta^2,\theta^2)\cdot\rho\nabla\rho + 2D_2\lambda(\rho^2,\eta^2,\theta^2)\cdot\eta\nabla\eta + 2D_3\lambda(\rho^2,\eta^2,\theta^2)\cdot\theta\nabla\theta$$

but on the homoclinic orbit $(x_1,y_1,x_2,y_2) = (\gamma_+(t),\dot\gamma_+(t))$ we have $\theta = \dot\varphi = 0$ and

$$\frac{x_1-a}{\rho_0(t_{a,b}^+ + t)} = u_+,\ \frac{y_1-b}{\rho_0(t_{a,b}^+ + t)} = v_+,\ \frac{x_2}{\dot\rho_0(t_{a,b}^+ + t)} = u_+,\ \frac{y_2}{\dot\rho_0(t_{a,b}^+ + t)} = v_+$$

so:

$$\nabla\rho = \begin{pmatrix} u_+ \\ v_+ \\ 0 \\ 0 \end{pmatrix}$$
$$\eta\nabla\eta = \dot\rho_0(t_{a,b}^+ + t)\begin{pmatrix} 0 \\ 0 \\ u_+ \\ v_+ \end{pmatrix}$$

and then:

$$\begin{aligned}\nabla\lambda(\gamma_+(t),\dot\gamma_+(t),0) &= 2D_1\lambda\left(\rho_0^2(t_{a,b}^+ + t), \dot\rho_0^2(t_{a,b}^+ + t), 0\right)\rho_0(t_{a,b}^+ + t)\begin{pmatrix} u_+ \\ v_+ \\ 0 \\ 0 \end{pmatrix}\\ &+2D_2\lambda\left(\rho_0^2(t_{a,b}^+ + t), \dot\rho_0^2(t_{a,b}^+ + t), 0\right)\dot\rho_0(t_{a,b}^+ + t)\begin{pmatrix} 0 \\ 0 \\ u_+ \\ v_+ \end{pmatrix}.\end{aligned}$$

Similarly on $(\gamma_-(t),\dot\gamma_-(t))$ we have $\theta = \dot\varphi = 0$ and

$$\frac{x_1-a}{\rho_0(t_{a,b}^- - t)} = u_-,\ \frac{y_1-b}{\rho_0(t_{a,b}^- - t)} = v_-,\ \frac{x_2}{\dot\rho_0(t_{a,b}^- - t)} = -u_-,\ \frac{y_2}{\dot\rho_0(t_{a,b}^- - t)} = -v_-$$

and then:

$$\begin{aligned}\lambda'(\gamma_-(t),\dot\gamma_-(t),0)^* &= 2D_1\lambda\left(\rho_0^2(t_{a,b}^- - t), \dot\rho_0^2(t_{a,b}^- - t), 0\right)\rho_0(t_{a,b}^- - t)\begin{pmatrix} u_- \\ v_- \\ 0 \\ 0 \end{pmatrix}\\ &-2D_2\lambda\left(\rho_0^2(t_{a,b}^- - t), \dot\rho_0^2(t_{a,b}^- - t), 0\right)\dot\rho_0(t_{a,b}^- - t)\begin{pmatrix} 0 \\ 0 \\ u_- \\ v_- \end{pmatrix}.\end{aligned}$$

We set, for simplicity,

$$\begin{aligned}\hat\lambda(t) &= \lambda\left(\rho_0^2(t), \dot\rho_0^2(t), 0\right)\\ \hat\lambda_1(t) &= D_1\lambda\left(\rho_0^2(t), \dot\rho_0^2(t), 0\right)\\ \hat\lambda_2(t) &= D_2\lambda\left(\rho_0^2(t), \dot\rho_0^2(t), 0\right)\end{aligned}$$

and

$$\begin{aligned}
\hat\lambda^{\pm}(t) &= \hat\lambda(t^{\pm}_{a,b} \pm t)\\
\hat\lambda_1^{\pm}(t) &= \hat\lambda_1(t^{\pm}_{a,b} \pm t)\\
\hat\lambda_2^{\pm}(t) &= \hat\lambda_2(t^{\pm}_{a,b} \pm t).
\end{aligned}$$

Then the linear variational system along $(\gamma_\pm(t), \dot\gamma_\pm(t))$ is ($+$ is for $t \geq 0$ and $-$ for $t < 0$)

$$\begin{pmatrix}\dot x_1\\ \dot y_1\\ \dot x_2\\ \dot y_2\end{pmatrix} = \begin{pmatrix}0 & \mathbb{I}\\ C_\pm(t) & D_\pm(t)\end{pmatrix}\begin{pmatrix}x_1\\ y_1\\ x_2\\ y_2\end{pmatrix} \tag{5.1}$$

where

$$\begin{aligned}
D_\pm(t) &= \pm 2\hat\lambda_2(t^{\pm}_{a,b} \pm t)\dot\rho_0(t^{\pm}_{a,b} \pm t)\begin{pmatrix} f_x(\gamma_\pm(t))u_\pm & f_x(\gamma_\pm(t))v_\pm\\ f_y(\gamma_\pm(t))u_\pm & f_y(\gamma_\pm(t))v_\pm\end{pmatrix}\\
&= \pm 2\hat\lambda_2(t^{\pm}_{a,b} \pm t)\dot\rho_0(t^{\pm}_{a,b} \pm t) f'(\gamma_\pm(t))^*(u_\pm,\ v_\pm)
\end{aligned}$$

and

$$\begin{aligned}
C_\pm(t) &= 2\hat\lambda_1(t^{\pm}_{a,b} \pm t)\rho_0(t^{\pm}_{a,b} \pm t)\begin{pmatrix} f_x(\gamma_\pm(t))u_\pm & f_x(\gamma_\pm(t))v_\pm\\ f_y(\gamma_\pm(t))u_\pm & f_y(\gamma_\pm(t))v_\pm\end{pmatrix} + \hat\lambda(t^{\pm}_{a,b} \pm t)H_f(\gamma_\pm(t))\\
&= 2\hat\lambda_1(t^{\pm}_{a,b} \pm t)\rho_0(t^{\pm}_{a,b} \pm t) f'(\gamma_\pm(t))^*(u_\pm,\ v_\pm) + \hat\lambda(t^{\pm}_{a,b} \pm t)H_f(\gamma_\pm(t)).
\end{aligned}$$

From (2.2) we know that

$$f'(\gamma_\pm(t))^* = 2\rho_0(t^{\pm}_{a,b} \pm t)F'(\rho_0^2(t^{\pm}_{a,b} \pm t))\left(\begin{smallmatrix}u_\pm\\ v_\pm\end{smallmatrix}\right)$$

and hence:

$$\begin{aligned}
D_\pm(t) &= \pm 4\hat\lambda_2(t^{\pm}_{a,b} \pm t)\rho_0(t^{\pm}_{a,b} \pm t))\dot\rho_0(t^{\pm}_{a,b} \pm t))F'(\rho_0^2(t^{\pm}_{a,b} \pm t))\begin{pmatrix} u_\pm^2, & u_\pm v_\pm\\ u_\pm v_\pm & v_\pm^2\end{pmatrix}\\
&= 2\hat\lambda_2(t^{\pm}_{a,b} \pm t)\tfrac{d}{dt}[F(\rho_0^2(t^{\pm}_{a,b} \pm t))]\begin{pmatrix} u_\pm^2, & u_\pm v_\pm\\ u_\pm v_\pm & v_\pm^2\end{pmatrix}.
\end{aligned}$$

As a consequence

$$D_\pm(t) = D_\pm(t)^* \quad \text{and similarly} \quad C_\pm(t) = C_\pm(t)^*. \tag{5.2}$$

So the adjoint system to (5.1) is:

$$\begin{pmatrix}\dot x_1\\ \dot y_1\\ \dot x_2\\ \dot y_2\end{pmatrix} = -\begin{pmatrix}0 & C_\pm(t)\\ \mathbb{I} & D_\pm(t)\end{pmatrix}\begin{pmatrix}x_1\\ y_1\\ x_2\\ y_2\end{pmatrix}.$$

with the impact condition (3.5) (and $w = (x_1, y_1, x_2, y_2)^*$).

For sake of simplicity we set:

$$\begin{aligned}
M_\pm &= \begin{pmatrix} u_\pm^2 & u_\pm v_\pm\\ u_\pm v_\pm & v_\pm^2\end{pmatrix}\\
d(t) &= \hat\lambda_2(t)\tfrac{d}{dt}[F(\rho_0^2(t))]\\
c(t) &= 4\hat\lambda_1(t)\rho_0^2(t)F'(\rho_0^2(t))\\
d_\pm(t) &= \pm 2d(t^{\pm}_{a,b} \pm t)\\
c_\pm(t) &= c(t^{\pm}_{a,b} \pm t)
\end{aligned}$$

then

$$D_\pm(t) = \pm 2d(t^\pm_{a,b} \pm t)M_\pm = d_\pm(t)M_\pm$$

$$C_\pm(t) = c(t^\pm_{a,b} \pm t)M_\pm + \hat\lambda(t^\pm_{a,b} \pm t)H_f(\gamma_\pm(t)) = c_\pm(t)M_\pm + \hat\lambda^\pm(t)H_f(\gamma_\pm(t)).$$

Let us write

$$\begin{pmatrix}\psi_1^\pm(t)\\ \psi_2^\pm(t)\end{pmatrix} = \psi(t) = X_\pm^{-1}(t)^* \begin{pmatrix}0\\0\\ \pm u_\pm\\ \pm v_\pm\end{pmatrix}.$$

Then

$$\begin{cases} \dot\psi_1^\pm(t) = -C_\pm(t)\psi_2^\pm(t)\\ \dot\psi_2^\pm(t) = -d_\pm(t)M_\pm\psi_2^\pm(t) - \psi_1^\pm(t)\\ \psi_1^\pm(0) = 0,\ \psi_2^\pm(0) = \pm\left(\begin{smallmatrix}u_\pm\\ v_\pm\end{smallmatrix}\right). \end{cases}$$

Suppose, for the moment, that $C_\pm(t)$ and $M_\pm$ commute, that is

$$C_\pm(t)M_\pm = M_\pm C_\pm(t), \tag{5.3}$$

then we see that

$$\begin{pmatrix}M_\pm\psi_1^\pm(t)\\ M_\pm\psi_2^\pm(t)\end{pmatrix}$$

satisfies the same equation and hence:

$$\begin{matrix}\psi_1^\pm(t) = M_\pm\psi_1^\pm(t)\\ \psi_2^\pm(t) = M_\pm\psi_2^\pm(t)\end{matrix}$$

or else:

$$\langle\psi_1^\pm(t), \left(\begin{smallmatrix}v_\pm\\ -u_\pm\end{smallmatrix}\right)\rangle = \langle\psi_2^\pm(t), \left(\begin{smallmatrix}v_\pm\\ -u_\pm\end{smallmatrix}\right)\rangle = 0.$$

This means that to determine $\psi(t)$ we do not need to study a 4-dimensional equation but a 2-dimensional equation that is the equation satisfied by the functions

$$\xi_1^\pm(t) := \langle\psi_1^\pm(t), \left(\begin{smallmatrix}u_\pm\\ v_\pm\end{smallmatrix}\right)\rangle; \quad \xi_2^\pm(t) := \langle\psi_2^\pm(t), \left(\begin{smallmatrix}u_\pm\\ v_\pm\end{smallmatrix}\right)\rangle.$$

From the expression of $C_\pm(t)$ we easily see that $C_\pm(t)M_\pm = M_\pm C_\pm(t)$ is equivalent to

$$H_f(\gamma_\pm(t))M_\pm = M_\pm H_f(\gamma_\pm(t))$$

But from (2.2) we derive:

$$H_f(\gamma_\pm(t)) = 2F'(\rho^2)\mathbb{I} + 4F''(\rho^2)\rho^2\begin{pmatrix}u_\pm^2 & u_\pm v_\pm\\ u_\pm v_\pm & v_\pm^2\end{pmatrix} = 2F'(\rho^2)\mathbb{I} + 4F''(\rho^2)\rho^2 M_\pm$$

(with $\rho = \rho_0(t^\pm_{a,b} \pm t)$) and then $H_f(\gamma_\pm(t))M_\pm = M_\pm H_f(\gamma_\pm(t))$ easily follows. Note that, being

$$\langle\psi_{1,2}^\pm(t), \left(\begin{smallmatrix}v_\pm\\ -u_\pm\end{smallmatrix}\right)\rangle = 0$$

we get, because of the orthonormality of the set $\{(u_\pm, v_\pm), (v_\pm, -u_\pm)\}$:

$$\psi_{1,2}^\pm(t) = \xi_{1,2}^\pm(t)\left(\begin{smallmatrix}u_\pm\\ v_\pm\end{smallmatrix}\right).$$

So, using also $M_\pm = M_\pm^*$ and $M_\pm\left(\begin{smallmatrix}u_\pm\\ v_\pm\end{smallmatrix}\right) = \left(\begin{smallmatrix}u_\pm\\ v_\pm\end{smallmatrix}\right)$: we see that $(\xi_1(t), \xi_2(t))$ satisfies the system in $\mathbb{R}^2$:

$$\begin{cases} \dot\xi_1^\pm(t) = -\langle C_\pm(t)\left(\begin{smallmatrix}u_\pm\\ v_\pm\end{smallmatrix}\right), \left(\begin{smallmatrix}u_\pm\\ v_\pm\end{smallmatrix}\right)\rangle\xi_2^\pm(t)\\ \dot\xi_2^\pm(t) = -\xi_1^\pm(t) - d_\pm(t)\xi_2^\pm(t)\\ \xi_1^\pm(0) = 0,\ \xi_2^\pm(0) = \pm 1. \end{cases} \tag{5.4}$$

Since

$$\langle C_\pm(t)\left(\begin{smallmatrix}u_\pm\\ v_\pm\end{smallmatrix}\right),\left(\begin{smallmatrix}u_\pm\\ v_\pm\end{smallmatrix}\right) = c_\pm(t)+\hat\lambda_\pm(t)\langle H_f(\gamma_\pm(t))\left(\begin{smallmatrix}u_\pm\\ v_\pm\end{smallmatrix}\right),\left(\begin{smallmatrix}u_\pm\\ v_\pm\end{smallmatrix}\right)\rangle$$

we can write (5.4) as

$$\left\{\begin{array}{l}\dot\xi_1^\pm(t) = -\left\{c_\pm(t)+\hat\lambda_\pm(t)\langle H_f(\gamma_\pm(t))\left(\begin{smallmatrix}u_\pm\\ v_\pm\end{smallmatrix}\right),\left(\begin{smallmatrix}u_\pm\\ v_\pm\end{smallmatrix}\right)\rangle\right\}\xi_2^\pm(t)\\ \dot\xi_2^\pm(t) = -\xi_1^\pm(t)-d_\pm(t)\xi_2^\pm(t)\\ \xi_1^\pm(0)=0,\ \xi_2^\pm(0)=\pm1.\end{array}\right. \tag{5.5}$$

or,

$$\left\{\begin{array}{l}\dot\xi_1^\pm(t) = -\left\{c_\pm(t)+2\hat\lambda^\pm(t)\left[F'(\rho_0^2(t_{a,b}^\pm\pm t))+2F''(\rho_0^2(t_{a,b}^\pm\pm t))\rho_0^2(t_{a,b}^\pm\pm t)\right]\right\}\xi_2^\pm(t)\\ \dot\xi_2^\pm(t) = -\xi_1^\pm(t)-d_\pm(t)\xi_2^\pm(t)\\ \xi_1^\pm(0)=0,\ \xi_2^\pm(0)=\pm1.\end{array}\right.$$

The equation adjoint to (5.5) (without initial conditions) is

$$\left\{\begin{array}{l}\dot\eta_1^\pm(t)=\eta_2(t)\\ \dot\eta_2(t)=\left\{c_\pm(t)+\hat\lambda^\pm(t)\langle H_f(\gamma_\pm(t))\left(\begin{smallmatrix}u_\pm\\ v_\pm\end{smallmatrix}\right),\left(\begin{smallmatrix}u_\pm\\ v_\pm\end{smallmatrix}\right)\rangle\right\}\eta_1^\pm(t)+d_\pm(t)\eta_2^\pm(t).\end{array}\right. \tag{5.6}$$

Since $(\dot\gamma_\pm(t),\ddot\gamma_\pm(t))$ is a solution of (5.1) satisfying the initial condition

$$(x_1(0),y_1(0),x_2(0),y_2(0)) = (\dot\gamma_\pm(0),\ddot\gamma_\pm(0)) = \mp\sqrt{2gF(0)}(u_\pm,v_\pm,0,0)$$

we see that $\left(\langle\dot\gamma_\pm(t),\left(\begin{smallmatrix}u_\pm\\ v_\pm\end{smallmatrix}\right)\rangle,\langle\ddot\gamma_\pm(t),\left(\begin{smallmatrix}u_\pm\\ v_\pm\end{smallmatrix}\right)\rangle\right)$ satisfies:

$$\left\{\begin{array}{l}\dot\eta_1(t)=\eta_2(t)\\ \dot\eta_2(t)=c_\pm(t)\eta_1(t)+\hat\lambda_\pm(t)\langle H_f(\gamma_\pm(t))\dot\gamma_\pm(t),\left(\begin{smallmatrix}u_\pm\\ v_\pm\end{smallmatrix}\right)\rangle+d_\pm(t)\eta_2(t)\end{array}\right.$$

but, since

$$\dot\gamma_\pm(t)=\langle\dot\gamma_\pm(t),\left(\begin{smallmatrix}u_\pm\\ v_\pm\end{smallmatrix}\right)\rangle\left(\begin{smallmatrix}u_\pm\\ v_\pm\end{smallmatrix}\right)+\langle\dot\gamma_\pm(t),\left(\begin{smallmatrix}v_\pm\\ -u_\pm\end{smallmatrix}\right)\rangle\left(\begin{smallmatrix}v_\pm\\ -u_\pm\end{smallmatrix}\right)$$

and

$$\langle H_f(\gamma_\pm(t))\left(\begin{smallmatrix}v_\pm\\ -u_\pm\end{smallmatrix}\right),\left(\begin{smallmatrix}u_\pm\\ v_\pm\end{smallmatrix}\right)\rangle=0$$

we see that $\left(\langle\dot\gamma_\pm(t),\left(\begin{smallmatrix}u_\pm\\ v_\pm\end{smallmatrix}\right)\rangle,\langle\ddot\gamma_\pm(t),\left(\begin{smallmatrix}u_\pm\\ v_\pm\end{smallmatrix}\right)\rangle\right)$ satisfies (5.6) with the initial conditions:

$$\eta_1^\pm(0)=\mp\sqrt{2gF(0)}\quad \eta_2^\pm(0)=0.$$

As a consequence the function:

$$\frac{1}{\sqrt{2gF(0)}}\begin{pmatrix}\langle\ddot\gamma_\pm(t),\left(\begin{smallmatrix}u_\pm\\ v_\pm\end{smallmatrix}\right)\rangle\\ -\langle\dot\gamma_\pm(t),\left(\begin{smallmatrix}u_\pm\\ v_\pm\end{smallmatrix}\right)\rangle\end{pmatrix}e^{-\int_0^t d_\pm(s)ds}$$

satisfies (5.5) (initial condition included). So:

$$\begin{pmatrix}\xi_1^\pm(t)\\ \xi_2^\pm(t)\end{pmatrix}=\frac{1}{\sqrt{2gF(0)}}e^{-\int_0^t d_\pm(s)ds}\begin{pmatrix}\langle\ddot\gamma_\pm(t),\left(\begin{smallmatrix}u_\pm\\ v_\pm\end{smallmatrix}\right)\rangle\\ -\langle\dot\gamma_\pm(t),\left(\begin{smallmatrix}u_\pm\\ v_\pm\end{smallmatrix}\right)\rangle\end{pmatrix}$$

and then:

$$\psi(t)=\begin{pmatrix}\xi_1^\pm(t)\left(\begin{smallmatrix}u_\pm\\ v_\pm\end{smallmatrix}\right)\\ \xi_2^\pm(t)\left(\begin{smallmatrix}u_\pm\\ v_\pm\end{smallmatrix}\right)\end{pmatrix}=\frac{1}{\sqrt{2gF(0)}}e^{-\int_0^t d_\pm(s)ds}\begin{pmatrix}\langle\ddot\gamma_\pm(t),\left(\begin{smallmatrix}u_\pm\\ v_\pm\end{smallmatrix}\right)\rangle\left(\begin{smallmatrix}u_\pm\\ v_\pm\end{smallmatrix}\right)\\ -\langle\dot\gamma_\pm(t),\left(\begin{smallmatrix}u_\pm\\ v_\pm\end{smallmatrix}\right)\rangle\left(\begin{smallmatrix}u_\pm\\ v_\pm\end{smallmatrix}\right)\end{pmatrix}.$$

From (2.20) we see that

$$\dot\gamma_\pm(t)=\pm\dot\rho_0(t\pm t_{a,b}^\pm)\left(\begin{smallmatrix}u_\pm\\ v_\pm\end{smallmatrix}\right),\qquad \ddot\gamma_\pm(t)=\ddot\rho_0(t\pm t_{a,b}^\pm)\left(\begin{smallmatrix}u_\pm\\ v_\pm\end{smallmatrix}\right)$$

and hence:

$$\psi(t) = \frac{1}{\sqrt{2gF(0)}} e^{-\int_0^t d_\pm(s)ds} \begin{pmatrix} \ddot{\gamma}_\pm(t) \\ -\dot{\gamma}_\pm(t) \end{pmatrix}.$$

Next we have

$$\hat{\lambda}_2(t) = -2\frac{F'(\rho_0(t)^2) + 2\rho_0(t)^2 F''(\rho_0(t)^2)}{1 + 4\rho_0(t)^2 F'(\rho_0(t)^2)^2}$$

and

$$d(t) = \hat{\lambda}_2(t)\frac{d}{dt}F(\rho_0(t)^2) = -\frac{4\rho_0(t)\rho_0'(t)F'(\rho_0(t)^2)\left[F'(\rho_0(t)^2) + 2\rho_0(t)^2 F''(\rho_0(t)^2)\right]}{1 + 4\rho_0(t)^2 F'(\rho_0(t)^2)^2}.$$

Hence

$$\begin{aligned} d_\pm(t) = \pm 2d(t_{a,b}^\pm \pm t) &= -\frac{8\rho_\pm(t)\rho_\pm'(t)F'(\rho_\pm(t)^2)\left[F'(\rho_\pm(t)^2) + 2\rho_\pm(t)^2 F''(\rho_\pm(t)^2)\right]}{1 + 4\rho_\pm(t)^2 F'(\rho_\pm(t)^2)^2} \\ &= -\frac{d}{dt}\ln[1 + 4\rho_\pm(t)^2 F'(\rho_\pm(t)^2)^2] \end{aligned}$$

for $\rho_\pm(t) = \rho_0(t_{(a,b)}^\pm \pm t)$. This gives

$$\mathrm{e}^{-\int_0^t d_\pm(s)ds} = 1 + 4\rho_\pm(t)^2 F'(\rho_\pm(t)^2)^2$$

and hence, using also the equality $1 + 4\rho_\pm(t)^2 F'(\rho_\pm(t)^2)^2 = 1 + \|\nabla f(\gamma_\pm(t))\|^2$:

$$\psi(t) = \frac{1 + \|f'(\gamma_\pm(t))\|^2}{\sqrt{2gF(0)}} \begin{pmatrix} \ddot{\gamma}_\pm(t) \\ -\dot{\gamma}_\pm(t) \end{pmatrix}$$

that coincides with (3.3).

6 Conclusions

The main purpose of this paper is to introduce a new class of relatively simple chaotic impact system consisting in non–flat billiards. Thus we have studied the behavior of a particle of unitary mass moving on a cartesian surface $z = f(x, y)$ in $\mathbb{R}^3$ and (x, y) belongs to a convex domain $\bar{\Omega}$ with piecewise smooth boundary. The particle is subject to the gravity field and is reflected with respect to the normal axis when it hits the (smooth part of the) boundary that, in turns is subject to a small amplitude periodic (or - more generally - almost periodic) force. Due to the complexity of the problem we have considered radially symmetric functions with compact support in the interior of Ω. In such conditions we have proved the existence of a piecewise smooth homoclinic orbit for the unperturbed problem (when the boundary of Ω is frozen) consisting of three smooth parts. Since the time spent by the solutions near the middle part of the homoclinic orbit is almost the same, we have replaced the equation with an impact equation assuming that when a solution hits $\partial\Omega$ with a certain speed it is immediately sent to another point of $\partial\Omega$ with another speed. This reflection law has been explicitly computed by studying the flow near the middle part of the homoclinic orbit. Then we used a result in [2] concerning chaotic behavior of impact system to construct the Melnikov function for such an impact dynamical system. To clarify the result we applied it to two concrete situations, the first when we have a moving boundary and the second (to show the wider applicability of the result) when the gravity varies periodically. We have seen that chaotic behavior of the dynamical system appears generically in such situations and we have also studied an example with a concrete function $f(x, y)$.

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