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# Asymptotic stability for nonlinear Kirchhoff systems☆

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**Abstract**

We study the asymptotic stability for solutions of the nonlinear damped Kirchhoff system, with homogeneous Dirichlet boundary conditions, under fairly natural assumptions on the external force $f$ and the distributed damping $Q$. Then the results are extended to a more delicate problem involving also an internal dissipation of higher order, the so called strongly damped Kirchhoff system. Finally, the study is further extended to strongly damped Kirchhoff–polyharmonic systems, which model several interesting problems of the Woinowsky–Krieger type.


*Keywords:* Nonlinear damped Kirchhoff systems; Strongly damped Kirchhoff systems; Asymptotic stability

## 1. Introduction

In this paper we first investigate the asymptotic behavior of solutions of the following problem involving a damped nonlinear Kirchhoff wave system

$$\begin{cases} u_{tt} - M(\|Du\|^2)\Delta u + Q(t,x,u,u_t) + f(x,u) = 0 & \text{in } \mathbb{R}_0^+ \times \Omega, \\ u(t,x) = 0 & \text{on } \mathbb{R}_0^+ \times \partial\Omega, \end{cases} \tag{1.1}$$

where $u = (u_1, \ldots, u_N) = u(t,x)$ is the vectorial displacement, $N \geq 1$, $\mathbb{R}_0^+ = [0,\infty)$, $\Omega$ is a bounded domain of $\mathbb{R}^n$, $M$ is given by

$$M(\tau) = a + b\gamma\tau^{\gamma-1}, \quad \tau \geq 0 \tag{1.2}$$

with $a, b \geq 0$, $a + b > 0$ and $\gamma > 1$, and $\|\cdot\| = \|\cdot\|_{[L^2(\Omega)]^N}$. (The time interval $\mathbb{R}_0^+$ can be replaced by any time semi-line $[T,\infty)$, $T > 0$.)

System (1.1) is said to be *non-degenerate* when $a > 0$ and $b \geq 0$, while *degenerate* when $a = 0$ and $b > 0$. When $a > 0$ and $b = 0$, system (1.1) is the usual well-known semilinear wave system.

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Throughout the paper we assume

$$Q \in C(\mathbb{R}_0^+ \times \Omega \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N), \qquad f \in C(\Omega \times \mathbb{R}^N \to \mathbb{R}^N).$$

The function $Q$, representing a *nonlinear damping*, is always assumed to verify

$$(Q(t, x, u, v), v) \geq 0 \quad \text{for all arguments } t, x, u, v, \tag{1.3}$$

where $(\cdot, \cdot)$ is the inner product of $\mathbb{R}^N$. The most common suppressions of the vibrations of an elastic structure, represented by $Q$, are of passive viscous type and absorb vibration energy.

The *external force* $f$ is assumed to be derivable from a potential $F$, that is

$$f(x, u) = \partial_u F(x, u), \tag{1.4}$$

where $F \in C^1(\Omega \times \mathbb{R}^N \to \mathbb{R}_0^+)$ and $F(x, 0) = 0$. Moreover, we allow $(f(x, u), u)$ to take negative values, that is

$$(f(x, u), u) \geq -a\mu|u|^2 \quad \text{in } \Omega \times \mathbb{R}^N, \tag{1.5}$$

for some $\mu \in [0, \mu_0)$, where $\mu_0$ denotes the first eigenvalue of $-\Delta$ in $\Omega$, with zero Dirichlet boundary conditions. When either $\mu = 0$ or $a = 0$, that is when (1.1) is degenerate, then (1.5) reduces to the more familiar condition $(f(x, u), u) \geq 0$, namely $f$ is *of restoring type*. Even if we are in the vectorial case, with $N$ possibly greater than one, we consider general dampings $Q$ involving (1.3), see condition (AS) in Section 2 and also [15].

In the more delicate case in which $n \geq 3$ and $p > r$, where $r = 2n/(n-2)$ denotes here the Sobolev exponent of the space $W_0^{1,2}(\Omega)$, a further natural growth condition is assumed on $f$, see Section 2.

In [16] global existence is proved without imposing any bound on the exponent $p$ of the source term $f$, when $f$ does not depend on $t$ as in our setting. This also justifies the importance to consider the case $n \geq 3$ and $p > r$ for asymptotic stability. We refer the reader to [16] for a complete recent bibliography for wave equations also with nonlinear dampings.

Problem (1.1) models several interesting phenomena studied in mathematical physics. In the case $N = 1$ and $n = 1$ problem (1.1) describes the nonlinear vibrations of an elastic string. The original equation is

$$\varrho h u_{tt} - \left\{ p_o + \frac{\mathcal{E} h}{2L} \int_0^L |u_x|^2 \mathrm{d}x \right\} u_{xx} + \delta u_t + f(x, u) = 0 \tag{1.6}$$

for $t \geq 0$ and $0 < x < L$, where $u = u(t, x)$ is the lateral displacement at the space coordinate $x$ and the time $t$, $\mathcal{E}$ the Young modulus, $\varrho$ the mass density, $h$ the cross section area, $L$ the length, $p_0$ the initial axial tension, $\delta$ the resistance modulus and $f$ the external force, for which the existence of a potential F is redundant, since one can simply integrate $f(x, u)$ with respect to $u$. When $\delta = f = 0$, Eq. (1.6) was first introduced by Kirchhoff [8]. Further details and physical models described by Kirchhoff's classical theory can be found in [18]. For a somewhat related approach in the autonomous case, we refer to the paper [1], which is devoted to semilinear wave equations with linear damping.

A canonical example of (1.1), which we contemplate here, is given by the system in $\mathbb{R}_0^+ \times \Omega$ with

$$\begin{gathered} Q(t, x, v) = A_1(t, x)|v|^{m-2}v + A_2(t, x)|v|^{q-2}v, \quad f(x, u) = V_1(x)|u|^{\tilde{p}-2}u - V_2(x)u, \\ \tilde{p} > 1, \quad 2 \leq m < q \leq \max\{\tilde{p}, r\} \text{ if } n \geq 3, \end{gathered} \tag{1.7}$$

where $A_1, A_2 \in C(\mathbb{R}_0^+ \times \Omega \to \mathbb{R}^{N \times N})$ are semidefinite positive matrices, $V_1, V_2 \in C(\Omega \to \mathbb{R}_0^+)$, with $\sup_\Omega [V_1(x) + V_2(x)] < \infty$, $\inf_\Omega V_1(x) > 0$, and $\sup_\Omega V_2(x) \leq a\mu$ for some $\mu \in [0, \mu_0)$ – cf. Sections 2 and 6.

In the context of problem (1.1) the question of asymptotic stability is best considered by means of the natural energy associated with the solutions of (1.1), namely

$$Eu(t) = \frac{1}{2} \left\{ \int_\Omega \left( |u_t(t, x)|^2 + a|Du(t, x)|^2 \right) \mathrm{d}x + b \left( \int_\Omega |Du(t, x)|^2 \mathrm{d}x \right)^\gamma \right\} + \int_\Omega F(x, u(t, x)) \mathrm{d}x.$$

In particular the *rest field* $u(t, x) \equiv 0$ will be called *asymptotically stable in the mean*, or simply *asymptotically stable,* if and only if

$$\lim_{t \to \infty} Eu(t) = 0 \quad \text{for all solutions } u = u(t, x) \text{ of (1.1)}.$$

In this formulation we have tacitly assumed that the solutions in question are classical, but for an adequate and useful theory one must actually consider solutions in a wider class of functions.

Indeed, one of the goals of this paper is to formulate a rational definition of solution for (1.1) which is independent of detailed properties of the functions $f$ and $Q$, and at the same time provides a useful framework for the study of asymptotic stability. This we do in Section 2. Section 3 is devoted to our main asymptotic stability result, Theorem 3.1, which is based on the a priori existence of a suitable auxiliary function $k = k(t)$, which was first introduced by Pucci and Serrin in [13]. With appropriate choices of the function $k$ we can obtain a number of useful special cases of Theorem 3.1, see Section 6.

Further applications to more general problems than (1.1) are given in Sections 4 and 5. In particular, in Section 4 we study the most delicate problem

$$\begin{cases} u_{tt} - M(\|Du\|^2)\Delta(u + \varrho(t)u_t) + Q(t, x, u, u_t) + f(x, u) = 0 & \text{in } \mathbb{R}_0^+ \times \Omega, \\ u(t, x) = 0 & \text{on } \mathbb{R}_0^+ \times \partial\Omega, \end{cases} \tag{1.8}$$

where $\varrho \in L^1_{\text{loc}}(\mathbb{R}_0^+ \to \mathbb{R}_0^+)$, which involves higher dissipation terms very interesting from an applicative point of view and of course includes the model (1.1) when $\varrho \equiv 0$, that is no higher dissipation terms are involved. Indeed, the expression $a\Delta(\varrho(t)u_t)$, involved in the second term of (1.8), represents the internal material damping of Kelvin–Voigt type of the body structure. For a detailed physical discussion the reader is referred to [12,7] as well as the references therein. However, it is worth noting that, in addition to the distributed damping $Q$, an internal damping mechanism is always present, even if small, in real materials as long as the system vibrates, see [6, Chapter 4, Dynamical Mechanical Properties, page 73]. For a further physical discussion on the common use in nonlinear acoustics, as well as in several other natural and industrial applications, of the dissipation higher-order term $a\Delta(\varrho(t)u_t)$, similar to the classical stress tensor describing a Stokesian fluid, we refer to [4]. Finally, we mention [3] for a model describing nonlinear viscoelastic materials with short memory in the special scalar case of (1.8), when $\varrho \equiv 1$, $Q \equiv 0$ and $f \equiv 0$.

In Section 5 we extend the study of Section 3 to the strongly damped Kirchhoff-polyharmonic systems in $\mathbb{R}_0^+ \times \Omega$

$$u_{tt} + (-\Delta)^L(\varrho(t)u_t) + M(\|\mathcal{D}_L u\|^2)(-\Delta)^L u + Q(t, x, u, u_t) + f(x, u) = 0, \quad L \geq 1, \tag{1.9}$$

which includes the model (1.1) when $L = 1$ and $\varrho \equiv 0$, and when $L \geq 2$

$$u_{tt} + (-\Delta)^L(u + \varrho(t)u_t) + M(\|\mathcal{D}_{L-1} u\|^2)(-\Delta)^{L-1} u + Q(t, x, u, u_t) + f(x, u) = 0, \tag{1.10}$$

both under the Dirichlet boundary conditions on $\mathbb{R}_0^+ \times \partial\Omega$

$$u(t, x) = 0, \qquad Du(t, x) = 0, \ldots, D^{2(L-1)}u(t, x) = 0, \tag{1.11}$$

where as above $\varrho \in L^1_{\text{loc}}(\mathbb{R}_0^+ \to \mathbb{R}_0^+)$, $M$ is given in (1.2),

$$\mathcal{D}_L u = \begin{cases} D\Delta^j u & \text{if } L = 2j + 1 \\ \Delta^j u & \text{if } L = 2j \end{cases} \quad \text{and} \quad n > 2L. \tag{1.12}$$

When $1 < p \leq r = 2n/(n - 2L)$, for the system (1.9) the source term $f$ is assumed to satisfy the corresponding condition (1.5), where now $\mu_0$ denotes the first eigenvalue of $(-\Delta)^L$ in $\Omega$, with zero Dirichlet boundary conditions, while for (1.10) it verifies

$$(f(x, u), u) \geq -\mu|u|^2 \quad \text{in } \Omega \times \mathbb{R}^N, \tag{1.13}$$

again for some $\mu \in [0, \mu_0)$. Finally, when $p > r$ for both systems $f$ is assumed as before to satisfy a further growth condition.

When $L = 2$ problems (1.9) and (1.10) are largely studied in the literature in several simplified subcases. However, the original physical models governed by (1.9) and (1.10) are vibrating beams of the Woinowsky–Krieger type, with internal material damping term of the Kelvin–Voigt type and a nonlinear damping $Q$ effective in $\Omega$.

The special brief Section 6 is devoted to simple consequences of the main results, which apply in the usual subcases of the systems considered here and somehow clarify the key asymptotic assumptions on the auxiliary function $k$. Even the special cases considered in Section 6 extend in several directions the most recent asymptotic stability results for damped systems, cf. i.e. [5,9,17] and also [2].

The primary consideration of this paper is the asymptotic stability of solutions of (1.1) and (1.8)–(1.10). Here we follow the approach of Pucci and Serrin in [13–15], where the authors treated the same subject for damped wave systems. Accordingly, we shall not be concerned with the problem of existence of solutions, which many authors have studied in some particular special cases. See, for example, [10,11] and the references cited therein. The reader is referred to this literature for further details.

## 2. Preliminaries

We first introduce the elementary bracket pairing in $\Omega \subset \mathbb{R}^n$,

$$\langle \varphi, \psi \rangle \equiv \int_\Omega (\varphi, \psi) \mathrm{d}x,$$

provided that $(\varphi, \psi) \in L^1(\Omega)$. We consider for simplicity

$$\mathbf{L}^\rho(\Omega) = [L^\rho(\Omega)]^N, \qquad X = [W_0^{1,2}(\Omega)]^N,$$

where $\rho > 1$, these spaces being endowed respectively with the natural norms

$$\|\varphi\|_\rho = \left(\int_\Omega |\varphi|^\rho\right)^{1/\rho}, \qquad \|D\varphi\| = \left(\int_\Omega |D\varphi|^2 \mathrm{d}x\right)^{1/2} = \left(\int_\Omega \sum_{i=1}^n |D_i\varphi|^2 \mathrm{d}x\right)^{1/2}.$$

We also put

$$\langle D\varphi, D\psi \rangle = \int_\Omega \sum_{i=1}^n (D_i\varphi, D_i\psi) \mathrm{d}x \quad \text{for all } \varphi, \psi \in X,$$

so in particular $\langle D\varphi, D\varphi \rangle = \|D\varphi\|^2$. Now define

$$K' = C(\mathbb{R}_0^+ \to X) \cap C^1(\mathbb{R}_0^+ \to \mathbf{L}^2(\Omega)) \quad \text{and} \quad K = \{\phi \in K' : E\phi \text{ is locally bounded on } \mathbb{R}_0^+\},$$

where $E\phi$ is the *total energy of the field* $\phi$, that is

$$E\phi = E\phi(t) = \frac{1}{2}\left(\|\phi_t\|^2 + a\|D\phi\|^2 + b\|D\phi\|^{2\gamma}\right) + \mathscr{F}\phi,$$

and $\mathscr{F}\phi$, the *potential energy of the field*, is given by

$$\mathscr{F}\phi = \mathscr{F}\phi(t) = \int_\Omega F(x, \phi(t, x)) \mathrm{d}x.$$

In writing $E\phi$ and $\mathscr{F}\phi$ we make the tacit agreement that $\mathscr{F}\phi$ is *well-defined*, namely that $F(\cdot, \phi(t, \cdot)) \in L^1(\Omega)$ for all $t \in \mathbb{R}_0^+$.

Since $\phi \in K$ the quantities $\phi_t, D\phi(t) \in \mathbf{L}^2(\Omega)$ for each $t \in \mathbb{R}_0^+$. Of course $\|\phi_t\|, \|D\phi\| \in L^\infty_{\mathrm{loc}}(\mathbb{R}_0^+)$, being continuous functions of $t$, so that $E\phi$ is locally bounded on $\mathbb{R}_0^+$ if and only if $\mathscr{F}\phi \in L^\infty_{\mathrm{loc}}(\mathbb{R}_0^+)$. Hence an equivalent definition of $K$ is given by

$$K = \{\phi \in K' : \mathscr{F}\phi \text{ is locally bounded on } \mathbb{R}_0^+\}. \tag{2.1}$$

Our motivation for introducing the set $K$ is that a solution of (1.1) should, whatever else, be sought in a function space for which the total energy is well-defined and bounded on any finite interval, and $K$ has just this property.

The definition of $K$ moreover applies without reference to the external force condition (1.5), so that the definition of solution given below applies equally whether $f$ satisfies (1.5) or not. Of course $f$ must be derivable from a potential as in (1.4).

We can now give our principal definition: a *strong solution of* (1.1) is a function $u \in K$ satisfying the following two conditions:

(A) *Distribution identity*

$$\langle u_t, \phi\rangle]_0^t = \int_0^t \{\langle u_t, \phi_t\rangle - M(\|Du\|^2)\langle Du, D\phi\rangle - \langle Q(\tau, \cdot, u, u_t), \phi\rangle - \langle f(\cdot, u), \phi\rangle\}\mathrm{d}\tau$$

for all $t \in \mathbb{R}_0^+$ and $\phi \in K$.

(B) *Conservation law*

(i) $\mathscr{D}u := \langle Q(t, \cdot, u, u_t), u_t\rangle \in L^1_{\mathrm{loc}}(\mathbb{R}_0^+)$,

(ii) $t \mapsto Eu(t) + \int_0^t \mathscr{D}u(\tau)\mathrm{d}\tau$ is non-increasing in $\mathbb{R}_0^+$.

We emphasize that *condition* (B) *is an essential attribute of solution*. Indeed, standard existence theorems for (1.1) in the literature always yield solutions satisfying *both* (A) and (B) in the stronger form in which the function in (B)-(ii) *is assumed to be constant*. On the other hand (A) alone does not imply (B), even if the integrability condition (B)-(i) is assumed a priori (see [19]). Conditions (B)-(ii) and (1.3) imply, however, that *$Eu$ is non-increasing in* $\mathbb{R}_0^+$.

A remaining issue is to determine a category of functions $f$ and $Q$ for which the preceding definition is meaningful. In particular, it must be shown that

$$\langle f(\cdot, u), \phi(t, \cdot)\rangle, \langle Q(t, \cdot, u, u_t), \phi(t, \cdot)\rangle \in L^1_{\mathrm{loc}}(\mathbb{R}_0^+), \tag{2.2}$$

so that the right-hand integral in identity (A) will be well-defined. To obtain (2.2) observe first that if $u, \phi \in K$, then

$$u, \phi \in C(\mathbb{R}_0^+ \to L^r(\Omega)), \tag{2.3}$$

where $r = 2n/(n-2)$ is the Sobolev exponent for the space $X$ (or $r$ is any real number satisfying $r > 2$ if $n = 1, 2$, because $\Omega$ is bounded).

We make the following natural hypotheses on $f$ and $Q$, in the principal case $n \geq 2$

(H) *Conditions* (1.4) *and* (1.5) *hold and there exists an exponent* $p \geq 2$ *such that*

(a) $|f(x, u)| \leq \text{Const.}\ (1 + |u|^{p-1})$ *for all* $(x, u) \in \Omega \times \mathbb{R}^N$.

*Moreover, if* $n \geq 3$ *and* $p > r$, $f$ *verifies* (a) *and*

(b) $(f(x, u), u) \geq \kappa_1|u|^p - \kappa_2|u|^{1/p} - \kappa_3|u|^r$ *for all* $(x, u) \in \Omega \times \mathbb{R}^N$

*for appropriate constants* $\kappa_1 > 0, \kappa_2, \kappa_3 \geq 0$.

When $f \equiv 0$ then (H)-(a) holds for any fixed $p \in [2, r)$, so that (H)-(b) is unnecessary. Moreover, whenever $n = 1$ or $n = 2$, we fix $r$ as any real number with $r > p$, so that again (H)-(b) is unnecessary.

(AS) *Condition* (1.3) *holds and there are exponents* $m, q$ *satisfying*

$$2 \leq m < q \leq s, \quad s = \max\{p, r\}, \quad \textit{while } s = \infty \quad \textit{if } n = 2,$$

*where* $m'$ *and* $q'$ *are the Hölder conjugates of* $m$ *and* $q$, *and non-negative continuous functions* $d_1 = d_1(t, x)$, $d_2 = d_2(t, x)$, *such that for all arguments* $t, x, u, v$,

(a) $|Q(t, x, u, v)| \leq d_1(t, x)^{1/m}(Q(t, x, u, v), v)^{1/m'} + d_2(t, x)^{1/q}(Q(t, x, u, v), v)^{1/q'}$,

*and the following functions* $\delta_1$ *and* $\delta_2$ *are well-defined*

$$\delta_1(t) = \|d_1(t, \cdot)\|_{s/(s-m)}, \qquad \delta_2(t) = \begin{cases} \|d_2(t, \cdot)\|_{s/(s-q)}, & \textit{if } q < s, \\ \|d_2(t, \cdot)\|_\infty, & \textit{if } q = s. \end{cases}$$

*Moreover, there are functions* $\sigma = \sigma(t)$, $\omega = \omega(\tau)$ *such that*

(b) $(Q(t, x, u, v), v) \geq \sigma(t)\omega(|v|)$ *for all arguments* $t, x, u, v$,

*where* $\omega \in C(\mathbb{R}_0^+ \to \mathbb{R}_0^+)$ *is such that*

$$\omega(0) = 0, \qquad \omega(\tau) > 0 \quad \textit{for } 0 < \tau < 1, \qquad \omega(\tau) = \tau^2 \quad \textit{for } \tau \geq 1,$$

*while* $\sigma \geq 0$ *and* $\sigma^{1-\wp} \in L^1_{\mathrm{loc}}(\mathbb{R}_0^+)$ *for some exponent* $\wp > 1$.

When $f \equiv 0$, by the above remark it is necessary to take $s = r$ in (AS). This simplifies the proofs of the main lemmas, since in this case $K = K'$.

It is worth observing that $f$ in (1.7), with $\tilde{p} > 1$, $V_1, V_2 \in C(\Omega \to \mathbb{R}_0^+)$, with $\sup_\Omega [V_1(x) + V_2(x)] < \infty$, and $\sup_\Omega V_2(x) \le a\mu$ for some $\mu \in [0, \mu_0)$, satisfies (1.5). Moreover, (H)-(a) holds with $p = 2$ when $1 < \tilde{p} < 2$, while with $p = \tilde{p}$ when $\tilde{p} \ge 2$. When $n \ge 3$ and $\tilde{p} > r$, then $f$ verifies (H)-(b) again with $p = \tilde{p}$, $\kappa_2 = \kappa_3 = \sup_\Omega V_2(x)$, provided that $\inf_\Omega V_1(x) =: \kappa_1 > 0$.

While the damping function $Q(t, x, v) = d_1(t, x)|v|^{m-2}v + d_2(t, x)|v|^{q-2}v$, $d_1, d_2 \in C(\mathbb{R}_0^+ \times \Omega \to \mathbb{R}_0^+)$, with $m$, $q$, $d_1$, $d_2$ as in (AS), satisfies (AS)-(a). For the proof see Section 2 of [15]. Furthermore, (AS)-(b) holds with $\omega(\tau) = \tau^q$ for $\tau \in [0, 1]$, $\omega(\tau) = \tau^2$ for $\tau \ge 1$, and $\sigma(t) = \inf_\Omega \{d_1(t, x) + d_2(t, x)\}$, provided that it is assumed $\sigma^{1-\wp} \in L^1_{\mathrm{loc}}(\mathbb{R}_0^+)$ for some $\wp > 1$. For more general $Q$ see Section 6.

The conditions (2.2) are consequences of the assumptions (H) and (AS). Indeed, by (H)-(a) for all $u, \phi \in K$

$$|\langle f(\cdot, u), \phi(t, \cdot)\rangle| \le \text{Const.}(\|\phi\|_1 + \|u\|_p^{p-1} \cdot \|\phi\|_p),$$

so that $\langle f(\cdot, u), \phi(t, \cdot)\rangle$ is locally bounded on $\mathbb{R}_0^+$, whenever $u, \phi \in K$, since $\|\cdot\|_p$ of any function of $K$ is locally bounded in $\mathbb{R}_0^+$ either by the Sobolev embedding theorem when $1 < p \le r$ or by the main assumption (H)-(b) when $p > r$ and $n \ge 3$. Indeed, in the latter case for all $u \in K$

$$\begin{aligned} F(x, u) &= \int_0^1 (f(x, \tau u), u)\mathrm{d}\tau \ge \int_0^1 (\kappa_1 |u|^p \tau^{p-1} - \kappa_2 |u|^{1/p} \tau^{-1/p'} - \kappa_3 |u|^r \tau^{r-1})\mathrm{d}\tau \\ &= \frac{\kappa_1}{p}|u|^p - p\kappa_2 |u|^{1/p} - \frac{\kappa_3}{r}|u|^r, \end{aligned}$$

and, since $\kappa_1 > 0$, we then have

$$\|u(t, \cdot)\|_p^p \le \frac{p}{\kappa_1}\left(\mathscr{F}u(t) + p\kappa_2 |\Omega|^{1/p'} \|u(t, \cdot)\|_1^{1/p} + \frac{\kappa_3}{r}\|u(t, \cdot)\|^r\right). \tag{2.4}$$

Therefore also $\|u\|_p \in L^\infty_{\mathrm{loc}}(\mathbb{R}_0^+)$, since $\mathscr{F}u$ is locally bounded by the definition of $K$ and $u \in X$. This completes the proof of the claim in the case $p > r$ and $n \ge 3$.

Moreover

$$\langle Q(t, \cdot, u, u_t), \phi(t, \cdot)\rangle \in L^1_{\mathrm{loc}}(\mathbb{R}_0^+), \tag{2.5}$$

since $\delta_1, \delta_2 \in L^1_{\mathrm{loc}}(\mathbb{R}_0^+)$ by (AS)-(a) and $\mathscr{D}u \in L^1_{\mathrm{loc}}(\mathbb{R}_0^+)$ by (B)-(i), see Section 2 of [15]. Thus (2.2) holds and so the *distribution identity* (A) is well-defined.

Finally, if either $1 < p \le r$ or $n = 2$ in (H), then $K = K'$, since $\phi \in C(\mathbb{R}_0^+ \to L^p(\Omega))$, see Section 2 of [14]. The case $n = 1$ will be treated at the end of Sections 3 and 4.

## 3. Asymptotic stability for damped nonlinear Kirchhoff systems

We recall that in what follows $\mu_0$ is the first eigenvalue of $-\Delta$ in $\Omega$, with zero Dirichlet boundary conditions.

**Theorem 3.1.** *Let* (H) *and* (AS) *hold. Suppose there exists a function k satisfying either*

$$k \in CBV(\mathbb{R}_0^+ \to \mathbb{R}_0^+) \quad \textit{and} \quad k \notin L^1(\mathbb{R}_0^+) \quad \textit{or}, \tag{3.1}$$

$$k \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}_0^+ \to \mathbb{R}_0^+), \quad k \not\equiv 0 \quad \textit{and} \quad \lim_{t\to\infty} \frac{\int_0^t |k'(\tau)|\mathrm{d}\tau}{\int_0^t k(\tau)\mathrm{d}\tau} = 0. \tag{3.2}$$

*Assume finally*

$$\liminf_{t\to\infty} \mathscr{A}(k(t)) \left(\int_0^t k(\tau)\mathrm{d}\tau\right)^{-1} < \infty, \tag{3.3}$$

*where*

$$\mathscr{A}(k(t)) = \mathscr{B}(k(t)) + \left(\int_0^t \sigma^{1-\wp} k^\wp \mathrm{d}\tau\right)^{1/\wp}, \quad \mathscr{B}(k(t)) = \left(\int_0^t \delta_1 k^m \mathrm{d}\tau\right)^{1/m} + \left(\int_0^t \delta_2 k^q \mathrm{d}\tau\right)^{1/q}. \tag{3.4}$$

*Then along any strong solution u of* (1.1) *we have*

$$\lim_{t\to\infty} Eu(t) = 0 \quad \textit{and} \quad \lim_{t\to\infty} (\|u_t\| + \|Du\|) = 0. \tag{3.5}$$

The integral condition (3.3) prevents the damping term $Q$ being either too small (*underdamping*) or too large (*overdamping*) as $t \to \infty$ and was introduced by Pucci and Serrin in [15], see also [13].

When $N = 1$, or $Q$ is tame, that is

($\mathscr{T}$) $Q$ is *tame*, if *there exists* $\kappa \geq 1$ *such that*

$$|Q(t, x, u, v)| \cdot |v| \leq \kappa (Q(t, x, u, v), v) \quad \text{(automatic if } N = 1\text{)},$$

then condition (AS)-(a) is equivalent to

$$|Q(t, x, u, v)| \leq \text{Const.}\ \{d_1(t, x)|v|^{m-1} + d_2(t, x)|v|^{q-1}\}$$

(this can be proved exactly as in Remark 1 of Section 5 in [15]).

Before proving Theorem 3.1 we give two preliminary lemmas under conditions (H) and (AS)-(a) which make the definition of strong solution meaningful.

**Lemma 3.2.** *Let u be a strong solution of* (1.1). *Then the non-increasing energy function Eu verifies in* $\mathbb{R}_0^+$

$$Eu \geq \frac{1}{2}\|u_t\|^2 + \frac{a}{2}\left(1 - \frac{\mu}{\mu_0}\right)\|Du\|^2 + \frac{b}{2}\|Du\|^{2\gamma} \geq 0. \tag{3.6}$$

*Moreover*

$$\|u_t\|, \|Du\|, \|u\|_p, \|u\|_r, M(\|Du\|^2) \in L^\infty(\mathbb{R}_0^+), \quad \mathscr{D}u = \langle Q(t, x, u, u_t), u_t\rangle \in L^1(\mathbb{R}_0^+). \tag{3.7}$$

**Proof.** By (1.5) we have $F(x, u) \geq -a\mu|u|^2/2$, so that $\mathscr{F}u(t) \geq -a\mu\|u(t, \cdot)\|^2/2$. Hence by the definition of $E$ and the fact that $b \geq 0$

$$Eu \geq \frac{1}{2}\|u_t\|^2 + \frac{1}{2}a\|Du\|^2 - \frac{1}{2}a\mu\|u\|^2 + \frac{1}{2}b\|Du\|^{2\gamma},$$

so (3.6) follows at once by Poincaré's inequality.

The proof of the fact that $\|u_t\|, \|Du\|, \|u\|_r \in L^\infty(\mathbb{R}_0^+)$ and $\mathscr{D}u \in L^1(\mathbb{R}_0^+)$ is exactly as for Lemma 3.1 of [15]. Clearly when $p \leq r$ then also $\|u\|_p \in L^\infty(\mathbb{R}_0^+)$. While if $n \geq 3$ and $p > r$, using (H)-(b) we get (2.4) since $u \in K$. Therefore also $\|u\|_p \in L^\infty(\mathbb{R}_0^+)$, since $\mathscr{F}u$ is bounded above – actually also below by (1.5) – being $Eu(t) \leq Eu(0)$ and $\|u_t\|, \|Du\| \in L^\infty(\mathbb{R}_0^+)$, and in turn also $M(\|Du\|^2) \in L^\infty(\mathbb{R}_0^+)$. This completes the proof of (3.7). □

By (B)-(ii) and Lemma 3.2 it is clear that there exists $l \geq 0$ such that

$$\lim_{t\to\infty} Eu(t) = l. \tag{3.8}$$

**Lemma 3.3.** *Let u be a strong solution of* (1.1) *and suppose* $l > 0$ *in* (3.8). *Then there exists a constant* $\alpha = \alpha(l) > 0$ *such that on* $\mathbb{R}_0^+$

$$\|u_t\|^2 + a\|Du\|^2 + b\|Du\|^{2\gamma} + \langle f(\cdot, u), u\rangle \geq \alpha. \tag{3.9}$$

**Proof.** The proof relies on the principal ideas used for proving [14, Lemma 3.4] and [15, Lemma 3.4]. Since $Eu(t) \geq l$ for all $t \in \mathbb{R}_0^+$ it follows that

$$\|u_t\|^2 + a\|Du\|^2 + b\|Du\|^{2\gamma} \geq 2(l - \mathscr{F}u) \quad \text{on } \mathbb{R}_0^+.$$

Let

$$J_1 = \{t \in \mathbb{R}_0^+ : \mathscr{F}u(t) \leq l/2\}, \qquad J_2 = \{t \in \mathbb{R}_0^+ : \mathscr{F}u(t) > l/2\}.$$

For $t \in J_1$

$$\|u_t\|^2 + a\|Du\|^2 + b\|Du\|^{2\gamma} \geq l. \tag{3.10}$$

Denoting by $\mathscr{L}u$ the left-hand side of (3.9), we find, using (1.5) and (3.10), that in $J_1$

$$\mathscr{L}u \geq a\left(1-\frac{\mu}{\mu_0}\right)\|Du\|^2 + \|u_t\|^2 + b\|Du\|^{2\gamma} \geq \left(1-\frac{\mu}{\mu_0}\right)l + \frac{\mu}{\mu_0}(\|u_t\|^2 + b\|Du\|^{2\gamma})$$
$$\geq \left(1-\frac{\mu}{\mu_0}\right)l.$$

Before dividing the proof into two parts, we observe that

$$|\mathscr{F}u| \leq C_1(\|u\|_1 + \|u\|_p^p) \tag{3.11}$$

by (H)-(a), see [15]. Now we denote with $\lambda_\rho$ the Sobolev constant of the embedding $W_0^{1,2}(\Omega) \hookrightarrow L^\rho(\Omega)$, for all $1 \leq \rho \leq r$, that is,

$$\|u\|_\rho \leq \lambda_\rho \|Du\|, \tag{3.12}$$

where $\lambda_\rho = \lambda_r |\Omega|^{1/\rho - 1/r}$ and depends on $n$, $\rho$, $|\Omega|$.

*Case* 1. $p \leq r$. By (3.11) and (3.12) we have $|\mathscr{F}u| \leq C(\|Du\| + \|Du\|^p)$, consequently in $J_2$

$$\tfrac{1}{2}l < \mathscr{F}u(t) \leq 2C \begin{cases} \|Du(t,\cdot)\|, & \text{if } \|Du(t,\cdot)\| \leq 1, \\ \|Du(t,\cdot)\|^p, & \text{if } \|Du(t,\cdot)\| > 1, \end{cases} \tag{3.13}$$

for an appropriate constant $C > 0$, depending on $C_1$ given in (3.11), $\lambda_1$, $\lambda_p$ introduced in (3.12) and $p$. Hence

$$\|Du(t,\cdot)\| \geq \min\left\{\frac{l}{4C}, \left(\frac{l}{4C}\right)^{1/p}\right\} = C_2(l) > 0,$$

and in $J_2$

$$\mathscr{L}u \geq a\left(1-\frac{\mu}{\mu_0}\right)C_2^2(l) + bC_2^{2\gamma}(l).$$

Therefore (3.9) holds with

$$\alpha = \alpha(l) = \left(1-\frac{\mu}{\mu_0}\right)\min\{l, aC_2^2(l)\} + bC_2^{2\gamma}(l) > 0,$$

provided that either $a \neq 0$ or $J_2 \neq \emptyset$, being $a + b > 0$.

Now, if $a = 0$ and $J_2 = \emptyset$, then (1.5) reduces to $(f(x,u), u) \geq 0$, and so (3.9) holds with $\alpha = l > 0$.

*Case* 2. $n \geq 3$ and $p > r$. Using (3.11), (H)-(b) and Hölder's inequality, we have for $t \in J_2$

$$\frac{l}{2} < \mathscr{F}u(t) \leq C_1(\|u(t,\cdot)\|_1 + \|u(t,\cdot)\|_p^p)$$
$$\leq c_0\left(\langle f(\cdot, u(t,\cdot)), u(t,\cdot)\rangle + \kappa_1\|u(t,\cdot)\|_1 + \kappa_2|\Omega|^{1/p'}\|u(t,\cdot)\|_1^{1/p} + \kappa_3\|u(t,\cdot)\|_r^r\right),$$

where $c_0 = C_1/\kappa_1$, since $\kappa_1 > 0$. But $\|u\|_1 \leq |\Omega|^{1/r'}\|u\|_r \leq \lambda_r|\Omega|^{1/r'}\|Du\|$ by (3.12) and Hölder's inequality. Therefore, by (3.12)

$$\langle f(\cdot,u), u\rangle + c_1\|Du\| + c_2\|Du\|^{1/p} + c_3\|Du\|^r > l/2c_0,$$

where $c_1 = \kappa_1\lambda_r|\Omega|^{1/r'}$, $c_2 = \kappa_2\lambda_r^{1/p}|\Omega|^{1-1/pr} \geq 0$ and $c_3 = \kappa_3\lambda_r^r \geq 0$. Hence for $t \in J_2$ and $\langle f(\cdot, u(t,\cdot)), u(t,\cdot)\rangle \geq 0$, then

$$\text{either} \quad \langle f(\cdot, u(t,\cdot)), u(t,\cdot)\rangle \geq l/4c_0 \quad \text{or} \quad \|Du(t,\cdot)\| \geq c_4, \tag{3.14}$$

where $c_4 = c_4(l, c_0) > 0$ is an appropriate constant, arising when $c_1\|Du\| + c_2\|Du\|^{1/p} + c_3\|Du\|^r \geq l/4c_0$. On the other hand, if $t \in J_2$ and $\langle f(\cdot, u(t,\cdot)), u(t,\cdot)\rangle < 0$, then $\|Du(t,\cdot)\| \geq c_5$, where $c_5 \geq c_4$ is an appropriate number arising from $c_1\|Du\| + c_2\|Du\|^{1/p} + c_3\|Du\|^r > l/2c_0$. By (1.5) the conclusion (3.9) holds, with

$$\alpha = \min\left\{(1-\mu/\mu_0)l,\, a(1-\mu/\mu_0)c_5^2 + bc_5^{2\gamma},\, ac_4^2 + bc_4^{2\gamma},\, l/4c_0\right\} > 0,$$

since $l > 0$, $\mu \in [0, \mu_0)$, $c_0 > 0$, $c_5 \geq c_4 > 0$, and $a + b > 0$. This completes the proof. □

**Proof of Theorem 3.1.** Following the main ideas of the proof of [14, Theorem 3.1] and [15, Theorem 1], first we treat case (3.1) in the simpler situation in which $k$ is not only $CBV(\mathbb{R}_0^+)$, but also of class $C^1(\mathbb{R}_0^+)$. Suppose, for contradiction that $l > 0$ in (3.8). Define a second Lyapunov function by

$$V(t) = k(t)\langle u, u_t\rangle = \langle u_t, \phi\rangle, \qquad \phi = k(t)u.$$

Since $k \in C^1(\mathbb{R}_0^+)$ and $\phi_t = k'u + ku_t$, it is clear that $\phi \in K$. Hence, by the distribution identity (A) in Section 2, we have for any $t \geq T \geq 0$

$$\begin{aligned} V(\tau)]_T^t &= \int_T^t \{k'\langle u, u_t\rangle + 2k\|u_t\|^2 - k[\|u_t\|^2 + M(\|Du\|^2)\|Du\|^2 + \langle f(\cdot,u), u\rangle]\}\mathrm{d}\tau \\ &\quad - \int_T^t k\langle Q(\tau,\cdot,u,u_t), u\rangle \mathrm{d}\tau. \end{aligned} \tag{3.15}$$

We now estimate the right-hand side of (3.15). First

$$\sup_{\mathbb{R}_0^+} |\langle u(t,\cdot), u_t(t,\cdot)\rangle| \leq \sup_{\mathbb{R}_0^+} \|u(t,\cdot)\|\cdot\|u_t(t,\cdot)\| = U < \infty \tag{3.16}$$

by (3.7) of Lemma 3.2. Now, using Lemma 3.3

$$-\int_T^t k\{\|u_t\|^2 + M(\|Du\|^2)\|Du\|^2 + \langle f(\cdot,u), u\rangle\}\mathrm{d}\tau \leq -\alpha\int_T^t k\mathrm{d}\tau, \tag{3.17}$$

and by Lemmas 3.2 and 3.3 of [15]

$$-\int_T^t k\langle Q(\tau,\cdot,u,u_t), u\rangle\mathrm{d}\tau \leq \varepsilon_1(T)\mathscr{B}(k(t)), \tag{3.18}$$

$$\int_T^t k\|u_t\|^2\mathrm{d}\tau \leq \theta\int_T^t k\mathrm{d}\tau + \varepsilon_2(T)C(\theta)\left(\int_0^t \sigma^{1-\wp}k^{\wp}\mathrm{d}\tau\right)^{1/\wp}, \tag{3.19}$$

where $C(\theta) = \omega_\theta^{1/\wp'}$, $\omega_\theta = \sup\{\tau^2/\omega(\tau) : \tau \geq \sqrt{\theta/|\Omega|}\}$,

$$\varepsilon_1(T) = \left(\sup_{\mathbb{R}_0^+}\|u(t,\cdot)\|_s\right)\cdot\left[\left(\int_T^\infty \mathscr{D}u(\tau)\mathrm{d}\tau\right)^{1/m'} + \left(\int_T^\infty \mathscr{D}u(\tau)\mathrm{d}\tau\right)^{1/q'}\right], \tag{3.20}$$

and

$$\varepsilon_2(T) = \left(\sup_{\mathbb{R}_0^+}\|u_t(t,\cdot)\|^{2/\wp}\right)\cdot\left(\int_T^\infty \mathscr{D}u(\tau)\mathrm{d}\tau\right)^{1/\wp'}, \tag{3.21}$$

with $\varepsilon_1(T) = o(1)$ and $\varepsilon_2(T) = o(1)$ as $T \to \infty$ by (3.7) of Lemma 3.2. Thus, by (3.15) it follows that

$$V(\tau)]_T^t \leq U\int_T^t |k'|\mathrm{d}\tau + 2\theta\int_T^t k\mathrm{d}\tau + 2\varepsilon(T)C(\theta)\left(\int_0^t \sigma^{1-\wp}k^{\wp}\mathrm{d}\tau\right)^{1/\wp} - \alpha\int_T^t k\mathrm{d}\tau + \varepsilon(T)\mathscr{B}(k(t)),$$

where $\varepsilon(T) = \max\{\varepsilon_1(T), \varepsilon_2(T)\}$. By (3.3) there is a sequence $t_i \nearrow \infty$ and a number $\ell > 0$ such that

$$\mathscr{A}(k(t_i)) \leq \ell\int_0^{t_i} k\mathrm{d}\tau. \tag{3.22}$$

Choose $\theta = \theta(l) = \alpha/4$ and fix $T > 0$ sufficiently large so that

$$\varepsilon(T)[2C(\theta) + 1]\ell \leq \alpha/4, \tag{3.23}$$

being $\varepsilon(T) = o(1)$ as $T \to \infty$. Consequently, for $t_i \geq T$,

$$V(t_i) \leq U \int_T^{t_i} |k'| \mathrm{d}\tau + S(T) - \frac{\alpha}{4} \int_T^{t_i} k \mathrm{d}\tau, \tag{3.24}$$

where $S(T) = V(T) + \varepsilon(T)[2C(\theta) + 1]\ell \int_0^T k\mathrm{d}\tau$. Thus by (3.1) we get

$$\lim_{i\to\infty} V(t_i) = -\infty, \tag{3.25}$$

since $k' \in L^1(\mathbb{R}_0^+)$ being $k \in CBV(\mathbb{R}_0^+)$. On the other hand, by (3.16) and recalling that $k$ is bounded,

$$|V(t)| \leq \left(\sup_{\mathbb{R}_0^+} k\right) \|u(t,\cdot)\| \cdot \|u_t(t,\cdot)\| \leq \left(\sup_{\mathbb{R}_0^+} k\right) U \quad \text{for all } t \in \mathbb{R}_0^+.$$

This contradiction completes the first part of the proof.

We pass to the general case $k \in CBV(\mathbb{R}_0^+)$ but not $k \in C^1(\mathbb{R}_0^+)$, following Lemma A in [13]. Let $\bar{k} \in C^1(\mathbb{R}_0^+)$ and $G \subset \mathbb{R}_0^+$ be an open subset such that

$$\text{(i)}\quad 2k \geq \bar{k} \geq \begin{cases} k & \text{in } \mathbb{R}_0^+ \setminus G \\ 0 & \text{in } G \end{cases}; \qquad \text{(ii)}\quad \operatorname{Var} \bar{k} \leq 2\operatorname{Var} k; \qquad \text{(iii)}\quad \int_G k\mathrm{d}s \leq 1.$$

Clearly $\bar{k} \in CBV(\mathbb{R}_0^+)$ by (ii). We next prove that $\bar{k}$ satisfies (3.1) and (3.3). Note that since $k \notin L^1(\mathbb{R}_0^+)$ it is possible to find a value $T_1$ such that

$$\int_0^{T_1} k\mathrm{d}\tau \geq 2. \tag{3.26}$$

Considering $t \geq T_1$, by (i), (ii) and (3.26) we obtain

$$\int_0^t \bar{k}\mathrm{d}\tau \geq \int_{[0,t]\setminus G} k\mathrm{d}\tau \geq \int_0^t k\mathrm{d}\tau - \int_G k\mathrm{d}\tau \geq \int_0^t k\mathrm{d}\tau - 1 \geq \frac{1}{2}\int_0^t k\mathrm{d}\tau. \tag{3.27}$$

Hence $\bar{k}$ satisfies (3.1). Moreover, by (i) and (3.27), for all $t \geq T_1$

$$\mathscr{A}(\bar{k}(t)) \int_0^t k\mathrm{d}\tau \leq 4\mathscr{A}(k(t)) \int_0^t \bar{k}\mathrm{d}\tau,$$

where $k \mapsto \mathscr{A}(k)$ is defined in (3.4). This shows that $\bar{k}$ also satisfies (3.3).

The general case is therefore reduced to the situation when $k$ is smooth, and the proof is complete in case (3.1).

If $k$ verifies (3.2) we again proceed by contradiction, supposing $l > 0$ in (3.8) and defining the auxiliary function $V(t) = k(t)\langle u, u_t\rangle = \langle u_t, \phi\rangle$ as in the case (3.1). Now we observe that by (3.2) we still obtain $\phi_t = k'u + ku_t$, so that $\phi \in K$. Since Lemmas 3.2 and 3.3 continue to hold also in this setting, as well as Lemmas 3.2 and 3.3 of [15], from now on we follow the proof of case (3.1) until obtaining (3.24). By the definition of $V$ we now get

$$|V(t_i)| \leq Uk(t_i) \leq U\left\{k(0) + \int_0^{t_i} |k'|\mathrm{d}\tau\right\},$$

so that by (3.24)

$$\frac{\alpha}{4}\int_0^{t_i} k\mathrm{d}\tau \leq 2U\int_0^{t_i} |k'|\mathrm{d}\tau + S(T) + Uk(0). \tag{3.28}$$

Dividing (3.28) by $\int_0^{t_i} k\mathrm{d}\tau$, using (3.2) and the fact that $k \notin L^1(\mathbb{R}_0^+)$ by (3.2), we obtain again a contradiction letting $i \to \infty$. Obviously condition $(3.5)_2$ follows immediately from (3.6). □

**Corollary 3.4.** *Let* (H) *and* (AS) *hold. Suppose that any one of the following conditions is satisfied:*

(a) $\sigma \in CBV(\mathbb{R}_0^+) \setminus L^1(\mathbb{R}_0^+)$, $\sigma^{m-1}\delta_1 + \sigma^{q-1}\delta_2 \in L^\infty(\mathbb{R}_0^+)$;

(b) $\delta_1^{1/(1-m)} \in CBV(\mathbb{R}_0^+) \setminus L^1(\mathbb{R}_0^+)$, $\wp = m$, $\sigma^{1-m}/\delta_1$, $\delta_2\delta_1^{(q-1)/(1-m)} \in L^\infty(\mathbb{R}_0^+)$;

(c) $\delta_2^{1/(1-q)} \in CBV(\mathbb{R}_0^+) \setminus L^1(\mathbb{R}_0^+)$, $\wp = q$, $\sigma^{1-q}/\delta_2$, $\delta_1\delta_2^{(m-1)/(1-q)} \in L^\infty(\mathbb{R}_0^+)$;

(d) $(\delta_1 + \sigma^{1-m})^{1/(1-m)} \in CBV(\mathbb{R}_0^+) \setminus L^1(\mathbb{R}_0^+)$, $\wp = m$, $\delta_2(\delta_1 + \sigma^{1-m})^{(q-1)/(1-m)} \in L^\infty(\mathbb{R}_0^+)$;

(e) $(\delta_2 + \sigma^{1-q})^{1/(1-q)} \in CBV(\mathbb{R}_0^+) \setminus L^1(\mathbb{R}_0^+)$, $\wp = q$, $\delta_1(\delta_2 + \sigma^{1-q})^{(m-1)/(1-q)} \in L^\infty(\mathbb{R}_0^+)$.

*Then the assertion of Theorem* 3.1 *holds.*

**Proof.** It is enough to take in each case, respectively,

$$k = \sigma, \qquad k = \delta_1^{1/(1-m)}, \qquad k = \delta_2^{1/(1-q)}, \qquad k = (\delta_1 + \sigma^{1-m})^{1/(1-m)}, \qquad k = (\delta_2 + \sigma^{1-q})^{1/(1-q)},$$

and find that (3.1) and (3.3) are satisfied. □

**The case n = 1**

When $n = 1$ we have $K = K'$. Furthermore, assumptions (H) and (AS) can be weakened respectively to

(H)′ $|f(x,u)| \le g(u)$, $\qquad g \in C(\mathbb{R}^N \to \mathbb{R}_0^+)$,

*for all* $(x,u) \in \Omega \times \mathbb{R}^N$.

(AS)′ *Let* (AS) *be satisfied, with* $\delta_1(t) = \|d_1(t,\cdot)\|_1$ *and* $\delta_2(t) = \|d_2(t,\cdot)\|_1$.

Clearly, (H)′ holds whenever $f \in C(\overline{\Omega} \times \mathbb{R}^N \to \mathbb{R})$, or $f$ does not depend on $x$. The above choice for the functions $\delta_1$ and $\delta_2$ in (AS)′ is justified by the fact that $r$ can be taken arbitrarily large when $n = 1$.

The next lemma will let the definition of *strong solution* meaningful.

**Lemma 3.5.** *Let* (H)′ *and* (AS)′ *hold. Then for all* $u, \phi \in K$ *and* $t \in \mathbb{R}_0^+$ *it results in* $\mathscr{F}\phi$ *well-defined and locally bounded in* $\mathbb{R}_0^+$*, and*

$$|\langle f(\cdot,u), \phi(t,\cdot)\rangle| \le c_1 \|\phi(t,\cdot)\|_{L^\infty(\Omega)},$$

*where* $c_1 = c_1(t) = |\Omega| \sup_{w \in B(t)} g(w)$*, and* $B$ *is defined by* $B(t) = \{w \in \mathbb{R}^N : |w| \le \|u(t,\cdot)\|_{L^\infty(\Omega)}\}$*. Moreover,* $\langle f(\cdot,u), \phi(t,\cdot)\rangle \in L^\infty_{\text{loc}}(\mathbb{R}_0^+)$ *and* $\langle Q(t,\cdot,u,u_t), \phi(t,\cdot)\rangle \in L^1_{\text{loc}}(\mathbb{R}_0^+)$.

**Proof.** The proof of the first part of the lemma is given in Section 2 of [15], noting that for all $\phi \in K$

$$\|\phi(t,\cdot)\|_\infty \le \sqrt{|\Omega|/2} \cdot \|D\phi(t,\cdot)\| \in C(\mathbb{R}_0^+),$$

and so $\langle f(\cdot,u), \phi(t,\cdot)\rangle$ is locally bounded on $\mathbb{R}_0^+$, since $c_1$ is locally bounded on $\mathbb{R}_0^+$. To see that $\langle Q(t,\cdot,u,u_t), \phi(t,\cdot)\rangle \in L^1_{\text{loc}}(\mathbb{R}_0^+)$ we first note that by Hölder's inequality and (AS)′

$$\begin{aligned}\|Q(t,\cdot,u,u_t)\|_1 &\le \|d_1(t,\cdot)^{1/m}(Q(t,\cdot,u,u_t),u_t)^{1/m'}\|_1 + \|d_2(t,\cdot)^{1/q}(Q(t,\cdot,u,u_t),u_t)^{1/q'}\|_1 \\ &\le \delta_1^{1/m}\mathscr{D}u^{1/m'} + \delta_2^{1/q}\mathscr{D}u^{1/q'}\end{aligned}$$

so that again by Hölder's inequality and the fact that $\|\phi(t,\cdot)\|_\infty \in L^1_{\text{loc}}(\mathbb{R}_0^+)$ for all $t \in \mathbb{R}_0^+$

$$\begin{aligned}\int_0^t |\langle Q(\tau,\cdot,u,u_t), \phi(\tau,\cdot)\rangle| \mathrm{d}\tau \le \max_{\tau\in[0,t]} \|\phi(\tau,\cdot)\|_\infty \Bigg[ &\left(\int_0^t \delta_1(\tau)\mathrm{d}\tau\right)^{1/m} \left(\int_0^t Du(\tau)\mathrm{d}\tau\right)^{1/m'} \\ &+ \left(\int_0^t \delta_2(\tau)\mathrm{d}\tau\right)^{1/q} \left(\int_0^t Du(\tau)\mathrm{d}\tau\right)^{1/q'} \Bigg],\end{aligned}$$

which completes the proof. □

**Theorem 3.6** (*The Case* $n = 1$). *Assume* (H)$'$ *and* (AS)$'$. *Suppose that there exists a function k as in Theorem* 3.1. *Then along any strong solution u of* (1.1) *property* (3.5) *holds.*

**Proof.** Note first that Lemma 3.2 holds, where now in (3.7) the norm $\|\cdot\|_r$ is replaced by $\|\cdot\|_\infty$. Moreover, Lemma 3.3 is clearly valid, with a simplified proof, where (3.11) is replaced by $|\mathscr{F}u(t,\cdot)| \le c\|u(t,\cdot)\|_1$, for some appropriate constant $c > 0$, and so (3.13) by $|\mathscr{F}u(t,\cdot)| \le C\|Du(t,\cdot)\|$, again for some appropriate constant $C > 0$. Finally, (3.9) holds with

$$\alpha = \alpha(l) = (1 - \mu/\mu_0)\min\{l, a(l/2C)^2\} + b(l/2C)^{2\gamma} > 0.$$

The proof of asymptotic stability is now exactly the same as before taking into account Lemma 3.5. $\square$

## 4. Kirchhoff higher-order damping terms

In this section we consider the more delicate system (1.8), where $\varrho \in L^1_{\text{loc}}(\mathbb{R}^+_0 \to \mathbb{R}^+_0)$, and take

$$X = \left[W^{1,2}_0(\Omega)\right]^N, \qquad K' = C^1(\mathbb{R}^+_0 \to X),$$

and $K$ in the usual way as in (2.1). Moreover, again

$$E\phi = E\phi(t) = \frac{1}{2}\left(\|\phi_t\|^2 + a\|D\phi\|^2 + b\|D\phi\|^{2\gamma}\right) + \mathscr{F}\phi.$$

By a *strong solution* of (1.8) we mean a function $u \in K$ satisfying the following two conditions

(A) *Distribution identity* for all $t \in \mathbb{R}^+_0$ and $\phi \in K$

$$\langle u_t, \phi\rangle]_0^t = \int_0^t \{\langle u_t, \phi_t\rangle - M(\|Du\|^2)\langle Du, D\phi\rangle - \varrho(\tau)M(\|Du\|^2)\langle Du_t, D\phi\rangle - \langle Q(\tau,\cdot,u,u_t),\phi\rangle - \langle f(\cdot,u),\phi\rangle\}\mathrm{d}\tau.$$

(B) *Conservation Law*

(i) $\mathscr{D}u := \langle Q(t,\cdot,u,u_t), u_t\rangle \in L^1_{\text{loc}}(\mathbb{R}^+_0)$,

(ii) $t \mapsto Eu(t) + \int_0^t \{\mathscr{D}u(\tau) + \varrho(\tau)M(\|Du\|^2)\|Du_t\|^2\}\mathrm{d}\tau$ is non-increasing in $\mathbb{R}^+_0$.

It is easy to see that this definition is meaningful when hypotheses (H) and (AS)-(a) hold. Also in this new context $Eu$ is *non-increasing in* $\mathbb{R}^+_0$ and so (3.8) continue to hold for some $l \ge 0$.

**Theorem 4.1.** *Let the assumptions of Theorem* 3.1 *hold, with the only exception that* (3.3) *is replaced by*

$$\liminf_{t\to\infty}\left[\left(\int_0^t \varrho k^2 \mathrm{d}\tau\right)^{1/2} + \mathscr{A}(k(t))\right] \Big/ \int_0^t k\mathrm{d}\tau < \infty, \tag{4.1}$$

*where* $t \mapsto \mathscr{A}(k(t))$ *is given in* (3.4). *Then along any strong solution u of* (1.8) *property* (3.5) *holds.*

Before proving this result we observe that we must amplify the discussion already given in Section 3 and take into account the more delicate term $\tau \mapsto \varrho(\tau)M(\|Du\|^2)\langle Du_t, D\phi\rangle$ in (A) and (B), which makes the analysis more involved when we use (B) in the degenerate case $a = 0$. Lemma 3.2 holds also in this new context and so (3.8) is true for some $l \ge 0$. Moreover we still require two further lemmas under assumptions (H) and (AS)-(a).

**Lemma 4.2.** *The conclusions* (3.6) *and* (3.7) *of Lemma* 3.2 *continue to hold. Moreover, the Kirchhoff damped function*

$$\mathscr{K}_\varrho := \varrho M(\|Du\|^2)\|Du_t\|^2 \in L^1(\mathbb{R}^+_0).$$

*Furthermore* $\varrho\|Du_t\|^2 \in L^1(\mathbb{R}^+_0)$ *whenever either* (1.8) *is non-degenerate, that is* $a > 0$, *or*

$$\inf_{\mathbb{R}^+_0}[a + b\gamma\|Du(t,\cdot)\|^{2\gamma-2}] > 0.$$

**Proof.** By (B) and (1.3)

$$0 \le \int_0^t \{\mathscr{D}u(\tau) + \varrho(\tau)M(\|Du\|^2)\|Du_t\|^2\}\mathrm{d}\tau \le Eu(0) - Eu(t) \le Eu(0),$$

since $Eu \ge 0$ in $\mathbb{R}_0^+$ by Lemma 3.2, and so $\mathscr{K}_\varrho \in L^1(\mathbb{R}_0^+)$. The last part of the lemma follows at once. □

**Lemma 4.3.** *For all $t \ge T \ge 0$ we have*

$$\int_T^t \varrho(\tau)k(\tau)M(\|Du\|^2)|\langle Du_t, Du\rangle|\mathrm{d}\tau \le \varepsilon_3(T)\left(\int_T^t \varrho(\tau)k^2(\tau)\mathrm{d}\tau\right)^{1/2}, \tag{4.2}$$

*where $\varepsilon_3(T) = \mathcal{K}\left(\int_T^\infty \mathscr{K}_\varrho(t)\mathrm{d}t\right)^{1/2} \to 0$ as $T \to \infty$ and*

$$\mathcal{K} := \sup_{\mathbb{R}_0^+}\left(\|Du(t,\cdot)\| \cdot \sqrt{M(\|Du\|^2)}\right).$$

**Proof.** By (3.7) clearly $\mathcal{K} < \infty$, since $\gamma > 1$. Hence by integration from $T$ to $t$ and use of Hölder's inequality twice, we obtain

$$\begin{aligned}\int_T^t \varrho(\tau)k(\tau)M(\|Du\|^2)|\langle Du, Du_t\rangle|\mathrm{d}\tau &\le \mathcal{K}\int_T^t \varrho(\tau)k(\tau)\sqrt{M(\|Du\|^2)}\|Du_t(\tau,\cdot)\|\mathrm{d}\tau \\ &\le \varepsilon_3(T)\left(\int_T^t \varrho(\tau)k^2(\tau)\mathrm{d}\tau\right)^{1/2},\end{aligned}$$

where $\varepsilon_3(T) \to 0$ as $T \to \infty$ by Lemma 4.2. □

**Proof of Theorem 4.1.** Suppose for contradiction that $Eu(t)$ approaches a limit $l > 0$ as $t \to \infty$. As in the proof of Theorem 3.1 we first treat the case (3.1) when $k$ is also of class $C^1(\mathbb{R}_0^+)$. Consider the Lyapunov function

$$V(t) = \langle u_t, \phi\rangle, \quad \phi = k(t)u \in K.$$

Hence by the distribution identity (A) above, for any $t \ge T \ge 0$, we have

$$\begin{aligned}V(\tau)]_T^t = &\int_T^t \{k'\langle u, u_t\rangle + 2k\|u_t\|^2 - k[\|u_t\|^2 + M(\|Du\|^2)\|Du\|^2 + \langle f(\cdot,u),u\rangle]\}\mathrm{d}\tau \\ &- \int_T^t \varrho k M(\|Du\|^2)\langle Du, Du_t\rangle\mathrm{d}\tau - \int_T^t k\langle Q(\tau,\cdot,u,u_t),u\rangle\mathrm{d}\tau.\end{aligned} \tag{4.3}$$

We first estimate the right-hand side of (4.3). Clearly (3.16) holds. Moreover, Lemma 3.3 and Lemmas 3.2–3.3 of [15] continue to hold, so we have again the estimates (3.17)–(3.19). Now, applying (3.16), (3.17)–(3.19) and (4.2), from (4.3) we obtain

$$\begin{aligned}V(\tau)]_T^t \le\ &U\int_T^t |k'|\mathrm{d}\tau + 2\theta\int_T^t k\mathrm{d}\tau + 2\varepsilon_2(T)C(\theta)\left(\int_0^t \sigma^{1-\wp}k^\wp\mathrm{d}\tau\right)^{1/\wp} \\ &- \alpha\int_T^t k\mathrm{d}\tau + \varepsilon_3(T)\left(\int_0^t \varrho k^2\mathrm{d}\tau\right)^{1/2} + \varepsilon_1(T)\mathscr{B}(k(t)),\end{aligned} \tag{4.4}$$

where $\varepsilon_1(T)$ is defined in (3.20), $\varepsilon_2(T)$ in (3.21), $\varepsilon_3(T)$ in (4.2) and $k \mapsto \mathscr{B}(k)$ in (3.4). By (4.1) there is a sequence $t_i \nearrow \infty$ and a number $\ell > 0$ such that

$$\left(\int_0^{t_i} \varrho k^2\mathrm{d}\tau\right)^{1/2} + \mathscr{A}(k(t_i)) \le \ell\int_0^{t_i} k\mathrm{d}\tau. \tag{4.5}$$

Choose $\theta = \theta(l) = \alpha/4$ and fix $T > 0$ so large that (3.23) holds, where now $\varepsilon(T) = \max\{\varepsilon_1(T), \varepsilon_2(T), \varepsilon_3(T)\}$. Hence for all $t_i \ge T \ge 0$ again (3.24) holds, and so, arguing as before, we obtain (3.25) by (3.1). The proof of the general case, $k \in CBV(\mathbb{R}_0^+)$ but not $k \in C^1(\mathbb{R}_0^+)$, proceeds as in Theorem 3.1.

In the case (3.2), arguing as in Theorem 3.1 and using (4.5) in the place of (3.22), we get again (3.28), which gives the required contradiction by (3.2). □

Combining Theorem 7.2 of [14], with the results of [15], we introduce another weaker version of Theorem 4.1 when $k$ and $\varrho$ are related.

**Theorem 4.4.** *Assume problem* (1.8) *is non-degenerate, that is* $a > 0$. *Let* (H), (AS)-(a) *hold and suppose there exists a function* $k$ *verifying*

$$k(t) \leq \text{Const.}\, \varrho(t) \quad \textit{for all } t \textit{ sufficiently large}, \tag{4.6}$$

*and either* (3.1) *or* (3.2). *Suppose finally*

$$\liminf_{t\to\infty} \left[ \left( \int_0^t \varrho k^2 \mathrm{d}\tau \right)^{1/2} + \mathscr{B}(k(t)) \right] \Bigg/ \int_0^t k \mathrm{d}\tau < \infty, \tag{4.7}$$

*where* $k \mapsto \mathscr{B}(k)$ *is defined in* (3.4). *Then along any strong solution of* (1.8) *property* (3.5) *holds.*

**Proof.** We first consider the case in which $k$ satisfies (3.1). Observe that Lemma 3.2 continue to hold, so that again (3.8) holds for some $l \geq 0$. Hence we follow the proof of Theorem 4.1 until the derivation of (4.3). Since Lemmas 3.2, 3.3, 4.2 and 4.3 continue to hold also in this context, as well as Lemma 3.2 of [15], the estimates (3.16)–(3.18) and (4.2) are still valid. While (3.19) no longer holds and Lemma 3.3 of [15] must be replaced by the following argument (cf. [14, Lemma 7.3]). Since $u_t \in C(\mathbb{R}_0^+ \to X)$, by Poincaré's inequality $\mu_0 \|u_t\|^2 \leq \|Du_t\|^2$. Hence by (4.6) and $T$ sufficiently large, for all $t \geq T$ we have

$$\int_T^t k\|u_t\|^2 \mathrm{d}\tau \leq \text{Const.} \int_T^t \varrho \|Du_t\|^2 \mathrm{d}\tau \leq \varepsilon_4(T), \tag{4.8}$$

where $\varepsilon_4(T) = \text{Const.} \int_T^\infty \varrho \|Du_t\|^2 \mathrm{d}\tau \to 0$ as $T \to \infty$, by Lemma 4.2 since $a > 0$, in replacement of (3.19). Therefore, instead of (4.4) we obtain

$$V(\tau)]_T^t \leq U \int_T^t |k'| \mathrm{d}\tau + 2\varepsilon_4(T) - \alpha \int_T^t k \mathrm{d}\tau + \varepsilon_3(T) \left( \int_0^t \varrho k^2 \mathrm{d}\tau \right)^{1/2} + \varepsilon_1(T) \mathscr{B}(k(t)), \tag{4.9}$$

where $\varepsilon_1(T)$ is defined in (3.20) and $\varepsilon_3(T)$ in (4.2). Therefore, we obtain again (3.24), where now $t_i \nearrow \infty$ and $\ell > 0$ are such that

$$\left( \int_0^{t_i} \varrho k^2 \mathrm{d}\tau \right)^{1/2} + \mathscr{B}(k(t_i)) \leq \ell \int_0^{t_i} k \mathrm{d}\tau \tag{4.10}$$

by (4.7), $\varepsilon(T) = \max\{\varepsilon_1(T), \varepsilon_3(T), \varepsilon_4(T)\} \leq 3\alpha/4\ell$ for $T$ large, and finally $S(T) = V(T) + \varepsilon(T)\{2 + \ell \int_0^T k \mathrm{d}\tau\}$. Hence (3.25) gives the required contradiction. When $k \notin C^1(\mathbb{R}_0^+)$ the proof derives exactly as in Theorem 3.1.

In case (3.2), arguing as in Theorem 3.1 and using (4.8) in the place of (3.19) and (4.10) in the place of (3.22), we get again (3.28), so that the contradiction follows by (3.2). □

**Theorem 4.5.** *Let* (1.8) *be non-degenerate and let the assumptions of either Theorem* 4.1 *or of Theorem* 4.4 *hold, with the only exception that* $k \notin L^1(\mathbb{R}_0^+)$ *is now of class* $W^{1,1}_{\text{loc}}(\mathbb{R}_0^+) \cap L^\infty(\mathbb{R}_0^+)$ *and satisfies also*

$$|k'| \leq \text{Const.} \sqrt{\varrho k} \quad \textit{a.e. in } \mathbb{R}_0^+. \tag{4.11}$$

*Then* (3.5) *holds.*

**Proof.** It is clear that the proof of the regular case of (3.1) can be applied also in this setting. It is enough to observe as in [14] that the term $k'\langle u, u_t \rangle$ is now estimated in the following way

$$|k'\langle u, u_t \rangle| \leq \text{Const.} \sqrt{\varrho k} \|u\| \cdot \|u_t\| \leq \text{Const.} \sqrt{\varrho k} \|Du_t\|,$$

by (4.11) and Hölder's and Poincaré's inequalities. Hence by Hölder's inequality

$$\int_T^t |k'\langle u, u_t\rangle| \mathrm{d}\tau \le \text{Const.} \left(\int_T^t k \mathrm{d}\tau\right)^{1/2} \left(\int_T^t \varrho \|Du_t\|^2 \mathrm{d}\tau\right)^{1/2} \le \varepsilon_5(T)\left(1 + \int_T^t k\mathrm{d}\tau\right),$$

where $\varepsilon_5(T) = \text{Const.} \left(\int_T^\infty \varrho \|Du_t\|^2 \mathrm{d}\tau\right)^{1/2} \to 0$ as $T \to \infty$ by Lemma 4.2 since $a > 0$. Hence (4.4) and (4.9) become, respectively,

$$\begin{aligned} V(\tau)]_T^t \;&\le\; \varepsilon_5(T)\left(1 + \int_T^t k\mathrm{d}\tau\right) + 2\theta \int_T^t k\mathrm{d}\tau + 2\varepsilon_2(T)C(\theta)\left(\int_0^t \sigma^{1-\wp} k^{\wp} \mathrm{d}\tau\right)^{1/\wp} \\ &\quad - \alpha \int_T^t k\mathrm{d}\tau + \varepsilon_3(T)\left(\int_0^t \varrho k^2 \mathrm{d}\tau\right)^{1/2} + \varepsilon_1(T)\mathscr{B}(k(t)), \\ V(\tau)]_T^t &\le \varepsilon_5(T)\left(1 + \int_T^t k\mathrm{d}\tau\right) + 2\varepsilon_4(T) - \alpha \int_T^t k\mathrm{d}\tau + \varepsilon_3(T)\left(\int_0^t \varrho k^2\mathrm{d}\tau\right)^{1/2} + \varepsilon_1(T)\mathscr{B}(k(t)), \end{aligned}$$

where again the dominating term is $-\alpha \int_T^t k\mathrm{d}\tau$ by (3.1). Call $\ell_1$ and $\ell_2$ the numbers verifying (4.5) and (4.10), respectively, in place of $\ell$, and take $\theta = \theta(l) = 3\alpha/16$. Furthermore, in the first case, we fix $T > 0$ so large that $\varepsilon(T) \le 3\alpha/8\,[1 + \ell_1 + 2C(\theta)\ell_1]$ and $\varepsilon(T) \le 3\alpha/4(1 + \ell_2)$ in the second case, where now of course $\varepsilon(T) = \max\{\varepsilon_1(T), \varepsilon_2(T), \varepsilon_3(T), \varepsilon_4(T), \varepsilon_5(T)\} = o(1)$ as $T \to \infty$. Proceeding as in either Theorem 4.1 or Theorem 4.4, respectively, in both cases we obtain

$$V(t_i) \le S(T) - \frac{\alpha}{4}\int_T^{t_i} k\mathrm{d}\tau, \tag{4.12}$$

where $S(T) = V(T) + \varepsilon(T)\{1 + [1 + 2C(\theta)]\ell_1 \int_0^T k\mathrm{d}\tau\}$ in the first case, while $S(T) = V(T) + \varepsilon(T)\{3 + \ell_2 \int_0^T k\mathrm{d}\tau\}$ in the second case. Since $k \notin L^1(\mathbb{R}_0^+)$, by (4.12) we get (3.25) which gives a contradiction using (3.16) and the fact that $k$ is bounded. □

**Theorem 4.6** (*The Case* $n = 1$). *Assume* (H)$'$ *and* (AS)$'$. *Suppose there exists a function $k$ as in one of the Theorems* 4.1, 4.4 *or* 4.5. *Then along any strong solution $u$ of* (1.8) *property* (3.5) *holds.*

**Proof.** With the modifications and simplifications already shown in the proof of Theorem 3.6 the result of Theorems 4.1, 4.4 and 4.5 follow by Lemma 3.5, since also Lemmas 4.2 and 4.3 continue to hold without changes. □

## 5. Kirchhoff-polyharmonic systems

Consider in $\mathbb{R}_0^+ \times \Omega$ the strongly damped systems (1.9) for $L \ge 1$ and (1.10) for $L \ge 2$, with Dirichlet boundary conditions (1.11) on $\mathbb{R}_0^+ \times \partial\Omega$, where $\varrho \in L^1_{\text{loc}}(\mathbb{R}_0^+ \to \mathbb{R}_0^+)$, $M$ is given in (1.2) and the operator $\mathcal{D}_L$ is defined in (1.12). As explained in Section 1, for the system (1.9) we assume condition (H)-(a), where now $\mu_0$ denotes the first eigenvalue of $(-\Delta)^L$ in $\Omega$, with zero Dirichlet boundary conditions, while for the system (1.10) we assume (H)-(a) with (1.5) replaced by (1.13). In both cases we assume (H)-(b), with $r = 2n/(n-2L)$ and $n > 2L$ in order to simplify the discussion. For clarity, in this section we denote this assumption by (H)$_L$. Put

$$X = [W_0^{L,2}(\Omega)]^N, \qquad K' = C^1(\mathbb{R}_0^+ \to X),$$

while $K$ is taken as always, see (2.1). The *total energy* associated to the systems (1.9) and (1.10) are defined, respectively, by

$$\begin{aligned} E_1\phi &= E_1\phi(t) = \frac{1}{2}\left(\|\phi_t\|^2 + a\|\mathcal{D}_L\phi\|^2 + b\|\mathcal{D}_L\phi\|^{2\gamma}\right) + \mathscr{F}\phi, \\ E_2\phi &= E_2\phi(t) = \frac{1}{2}\left(\|\phi_t\|^2 + \|\mathcal{D}_L\phi\|^2 + a\|\mathcal{D}_{L-1}\phi\|^2 + b\|\mathcal{D}_{L-1}\phi\|^{2\gamma}\right) + \mathscr{F}\phi. \end{aligned}$$

Also in the definition of solution we give two different expressions of the *Distribution Identity*. More precisely, we define a *strong solution* of (1.9) and (1.10), with the boundary conditions (1.11), a function $u \in K$ satisfying, respectively, (A1) *Distribution identity*

$$\langle u_t, \phi\rangle]_0^t = \int_0^t \{\langle u_t, \phi_t\rangle - M(\|\mathcal{D}_L u\|^2)\langle \mathcal{D}_L u, \mathcal{D}_L \phi\rangle - \varrho(\tau)\langle \mathcal{D}_L u_t, \mathcal{D}_L \phi\rangle - \langle Q(\tau,\cdot,u,u_t),\phi\rangle - \langle f(\cdot,u),\phi\rangle\}\mathrm{d}\tau,$$

(A2) *Distribution identity*

$$\langle u_t, \phi\rangle]_0^t = \int_0^t \{\langle u_t, \phi_t\rangle - \langle \mathcal{D}_L u, \mathcal{D}_L \phi\rangle - M(\|\mathcal{D}_{L-1} u\|^2)\langle \mathcal{D}_{L-1} u, \mathcal{D}_{L-1} \phi\rangle - \varrho(\tau)\langle \mathcal{D}_L u_t, \mathcal{D}_L \phi\rangle - \langle Q(\tau,\cdot,u,u_t),\phi\rangle - \langle f(\cdot,u),\phi\rangle\}\mathrm{d}\tau,$$

for all $t \in \mathbb{R}_0^+$ and $\phi \in K$.

(B) *Conservation law*

(i) $\mathscr{D}u := \langle Q(t,\cdot,u,u_t),u_t\rangle \in L^1_{\mathrm{loc}}(\mathbb{R}_0^+)$,

(ii) $t \mapsto E_i u(t) + \int_0^t \{\mathscr{D}u(\tau) + \varrho(\tau)\|\mathcal{D}_L u_t\|^2\}\mathrm{d}\tau$ is non-increasing in $\mathbb{R}_0^+$, $\quad i = 1, 2$.

These definitions are meaningful under the hypotheses $(\mathrm{H})_L$ and (AS)-(a). Once again conditions (B)-(ii) and (1.3) imply that both $E_1 u$ and $E_2 u$ *are non-increasing in* $\mathbb{R}_0^+$.

As explained in the introduction, there are several relevant physical phenomena modeled by systems (1.9) and (1.10). The most studied problems arise when $L = 2$. Of course, (1.9), with the boundary conditions (1.11), reduces to the prototype problem (1.1) when $L = 1$ and $\varrho \equiv 0$.

In the next preliminary lemmas we assume $(\mathrm{H})_L$ and (AS)-(a).

**Lemma 5.1.** *Let u be a strong solution either of* (1.9) *or* (1.10). *Then the total energies $E_1u$, $E_2u$ satisfy in* $\mathbb{R}_0^+$

$$\begin{aligned} E_1 u &\geq \frac{1}{2}\|u_t\|^2 + \frac{a}{2}\left(1 - \frac{\mu}{\mu_0}\right)\|\mathcal{D}_L u\|^2 + \frac{b}{2}\|\mathcal{D}_L u\|^{2\gamma} \geq 0, \\ E_2 u &\geq \frac{1}{2}\left[\|u_t\|^2 + \left(1 - \frac{\mu}{\mu_0}\right)\|\mathcal{D}_L u\|^2\right] \geq 0. \end{aligned} \tag{5.1}$$

*Moreover,*

$$\|u_t\|, \|\mathcal{D}_L u\|, \|\mathcal{D}_{L-1} u\|, \|u\|_r \in L^\infty(\mathbb{R}_0^+), \tag{5.2}$$

*while*

$$\varrho\|\mathcal{D}_L u_t\|^2, \mathscr{D}u = \langle Q(t,\cdot,u,u_t),u_t\rangle \in L^1(\mathbb{R}_0^+). \tag{5.3}$$

**Proof.** The proof is almost exactly the same as for Lemma 3.2, provided that $r$ is the appropriate Sobolev exponent. Indeed, (5.1) follows directly from (1.5), (1.13) and the corresponding Poincaré inequality

$$\mu_0\|u\|^2 \leq \|\mathcal{D}_L u\|^2,$$

where $\mu_0$ is defined at the beginning of this section. By (5.1) it follows that $\|u_t\|$ and $\|\mathcal{D}_L u\|$ are in $L^\infty(\mathbb{R}_0^+)$, and so also $\|\mathcal{D}_{L-1} u\|$ by Theorem 4.4.1 of [20], while the fact that $\|u\|_r \in L^\infty(\mathbb{R}_0^+)$ follows from the Sobolev inequality. Both the conditions in (5.3) are deduced in the same manner, almost as in Section 4. More precisely using (B)-(ii)

$$0 \leq \int_0^t \left\{\mathscr{D}u(\tau) + \varrho(\tau)\|\mathcal{D}_L u_t\|^2\right\}\mathrm{d}\tau \leq Eu(0),$$

since $Eu \geq 0$ in $\mathbb{R}_0^+$, so (5.3) follows at once by (1.3). $\square$

By (B)-(ii) and Lemma 5.1 there exist $l_i \geq 0$, $i = 1, 2$, such that

$$\lim_{t\to\infty} E_i u(t) = l_i. \tag{5.4}$$

In the new models (1.9) and (1.10) the term $\tau \mapsto \varrho(\tau)\langle \mathcal{D}_L u_t, \mathcal{D}_L \phi\rangle$ in (A1) and (A2) needs to be evaluated with some care, since it corresponds to the dynamical strongly viscous effects of the body (*bar* when $L = 1$, and *beam* $L = 2$).

**Lemma 5.2.** *For all $t \geq T \geq 0$ we have*

$$\int_T^t \varrho(\tau)k(\tau)\langle \mathcal{D}_L u, \mathcal{D}_L u_t\rangle \mathrm{d}\tau \leq \varepsilon_6(T)\left(\int_T^t \varrho(\tau)k^2(\tau)\mathrm{d}\tau\right)^{1/2},$$

*where* $\varepsilon_6(T) = \left(\sup_{\mathbb{R}_0^+} \|\mathcal{D}_L u(t,\cdot)\|\right) \cdot \left(\int_T^\infty \varrho(\tau)\|\mathcal{D}_L u_t\|^2 \mathrm{d}\tau\right)^{1/2} \to 0$ *as* $T \to \infty$.

**Proof.** We proceed as in Lemma 4.3. First of all we observe that by Hölder's inequality and (5.2) in $\mathbb{R}_0^+$

$$|\langle \mathcal{D}_L u(t,\cdot), \mathcal{D}_L u_t(t,\cdot)\rangle| \leq \left(\sup_{\mathbb{R}_0^+} \|\mathcal{D}_L u(t,\cdot)\|\right) \|\mathcal{D}_L u_t\|.$$

Then by integration from $T$ and $t$ and another use of Hölder's inequality we obtain

$$\int_T^t \varrho(\tau)k(\tau)\langle \mathcal{D}_L u, \mathcal{D}_L u_t\rangle \mathrm{d}\tau \leq \varepsilon_6(T)\left(\int_T^t \varrho(\tau)k^2(\tau)\mathrm{d}\tau\right)^{1/2},$$

where $\varepsilon_6(T) \to 0$ as $T \to \infty$ by (5.3). □

**Lemma 5.3.** *Suppose $l_i > 0$, $i = 1, 2$, in (5.4). Then there exist $\alpha_i = \alpha_i(l_i) > 0$ such that in $\mathbb{R}_0^+$*

$$\|u_t\|^2 + a\|\mathcal{D}_L u\|^2 + b\|\mathcal{D}_L u\|^{2\gamma} + \langle f(\cdot,u), u\rangle \geq \alpha_1,$$
$$\|u_t\|^2 + \|\mathcal{D}_L u\|^2 + a\|\mathcal{D}_{L-1} u\|^2 + b\|\mathcal{D}_{L-1} u\|^{2\gamma} + \langle f(\cdot,u), u\rangle \geq \alpha_2.$$

**Proof.** For system (1.9) the proof is analogous as the one in Lemma 3.3, taking here into account the appropriate Sobolev exponent $r$, for the space $X = [W_0^{L,2}(\Omega)]^N$.

For system (1.10) in $J_1$ we obtain, arguing as in Lemma 3.3 and using (1.13),

$$\mathscr{L}u \geq \left(1 - \frac{\mu}{\mu_0}\right) l_2.$$

Following the proof of Lemma 3.3 now $\mathcal{D}_L u$ replaces $Du$ in (3.13) and $C$ depends on the corresponding constants $\lambda_1$ and $\lambda_p$ now arising from

$$\|u\|_\rho \leq \lambda_\rho \|\mathcal{D}_L u\|. \tag{5.5}$$

When $t \in J_2$ we again distinguish two cases depending on whether $1 < p \leq r$ or $p > r$. For $1 < p \leq r$ by (1.13) it results

$$\mathscr{L}u \geq \left(1 - \frac{\mu}{\mu_0}\right) C_2^2(l_2).$$

Therefore

$$\alpha_2 = \alpha_2(l_2) = \left(1 - \frac{\mu}{\mu_0}\right) \min\{l_2, C_2^2(l_2)\}.$$

Analogously, in the case $p > r$, using (5.5) we obtain (3.14) with $\mathcal{D}_L u$ in the place of $Du$. Thus we find

$$\alpha_2 = \alpha_2(l_2) = \min\left\{\left(1 - \frac{\mu}{\mu_0}\right) l_2, \left(1 - \frac{\mu}{\mu_0}\right) c_5^2(l_2), c_4^2(l_2), l_2/4c_0\right\},$$

with $c_0 = C_1/\kappa_1$, arising from (3.11) and $(\mathrm{H})_L$-(b). □

**Theorem 5.4.** *Let $(\mathrm{H})_L$ and (AS) hold. If $k$ is a function as in Theorem 4.1, then along any strong solution $u$ of either (1.9) or (1.10) we have*

$$\lim_{t\to\infty} E_i u(t) = 0 \quad \text{and} \quad \lim_{t\to\infty} (\|u_t\| + \|\mathcal{D}_L u\|) = 0. \tag{5.6}$$

**Proof.** As in the proof of Theorem 4.1 we first consider $k$ in (3.1) also of class $C^1(\mathbb{R}_0^+)$ and suppose for contradiction that $l_i > 0$ in (5.4).

Define $V_i(t) = k(t)\langle u, u_t\rangle = \langle u_t, \phi\rangle$, $\phi = k(t)u$, $i = 1, 2$, where we have denoted the fixed solution $u_i$ by $u$ for simplicity. Hence $\phi \in K$. From the distribution identities (A1) and (A2) we obtain for all $t \geq T \geq 0$

$$\begin{aligned} V_1(\tau)]_T^t &= \int_T^t \left\{k'\langle u, u_t\rangle + 2k\|u_t\|^2 - k[\|u_t\|^2 + M(\|\mathcal{D}_L u\|^2)\|\mathcal{D}_L u\|^2 + \langle f(\cdot, u), u\rangle]\right\} \mathrm{d}\tau \\ &\quad - \int_T^t \varrho k\langle \mathcal{D}_L u, \mathcal{D}_L u_t\rangle \mathrm{d}\tau - \int_T^t k\langle Q(\tau, \cdot, u, u_t), u\rangle \mathrm{d}\tau; \\ V_2(\tau)]_T^t &= \int_T^t \left\{k'\langle u, u_t\rangle + 2k\|u_t\|^2 - k[\|u_t\|^2 + \|\mathcal{D}_L u\|^2 + M(\|\mathcal{D}_{L-1} u\|^2)\|\mathcal{D}_{L-1} u\|^2 + \langle f(\cdot, u), u\rangle]\right\} \mathrm{d}\tau \\ &\quad - \int_T^t \varrho k\langle \mathcal{D}_L u, \mathcal{D}_L u_t\rangle \mathrm{d}\tau - \int_T^t k\langle Q(\tau, \cdot, u, u_t), u\rangle \mathrm{d}\tau. \end{aligned}$$

Lemmas 3.2 and 3.3 of [15] continue to hold in this context, so that it is possible to make use of (3.18) and (3.19), while (3.16) now holds by (5.2) of Lemma 5.1. Moreover, Lemmas 5.2 and 5.3 and the above formulas for $V_i$ yield

$$\begin{aligned} V_i(\tau)]_T^t &\leq U \int_T^t |k'| \mathrm{d}\tau + 2\theta \int_T^t k \mathrm{d}\tau + 2\varepsilon_2(T)C(\theta) \left(\int_0^t \sigma^{1-\wp} k^{\wp} \mathrm{d}\tau\right)^{1/\wp} \\ &\quad - \alpha_i \int_T^t k \mathrm{d}\tau + \varepsilon_6(T) \left(\int_0^t \varrho k^2 \mathrm{d}\tau\right)^{1/2} + \varepsilon_1(T)\mathscr{B}(k(t)), \end{aligned}$$

instead of (4.4), where $\varepsilon_1(T)$ is given in (3.20), $\alpha_i$ as in Lemma 5.3, $\varepsilon_2(T)$ in (3.21), while $\varepsilon_6(T)$ is given in Lemma 5.2.

From now on the proof is the same as in Theorem 4.1, where $\alpha$ and $\varepsilon_3(T)$ are replaced by $\alpha_i$ and $\varepsilon_6(T)$ respectively, and of course $\varepsilon(T) = \max\{\varepsilon_1(T), \varepsilon_2(T), \varepsilon_6(T)\}$. □

**Theorem 5.5.** *Let* (H)$_L$ *and* (AS)-(a) *hold. The assertion of Theorem* 5.4 *continue to hold when k is as either in Theorem* 4.4 *or in the second part of Theorem* 4.5.

In the relevant physical case $L = 2$, an interesting subcase of the system (1.10) when $\varrho \equiv 0$ is given again in $\mathbb{R}_0^+ \times \Omega$ by the system

$$u_{tt} + (-\Delta)^L u - M(\|Du\|^2)\Delta u + Q(t, x, u, u_t) + f(x, u) = 0, \quad L \geq 1 \tag{5.7}$$

with Dirichlet boundary conditions (1.11) on $\mathbb{R}_0^+ \times \partial\Omega$, where $\mathcal{D}_L$ denotes as usual the operator given in (1.12). Here $X = [W_0^{L,2}(\Omega)]^N$ as above, but

$$K' = C(\mathbb{R}_0^+ \to X) \cap C^1(\mathbb{R}_0^+ \to L^2(\Omega)),$$

as in Section 3, and $K$ as in (2.1). The total energy is now given by

$$E\phi = E\phi(t) = \frac{1}{2}\left(\|\phi_t\|^2 + \|\mathcal{D}_L\phi\|^2 + a\|D\phi\|^2 + b\|D\phi\|^{2\gamma}\right) + \mathscr{F}\phi.$$

By a *strong solution* of (5.7) we mean a function $u \in K$ satisfying
(A) *Distribution identity*: for all $t \in \mathbb{R}_0^+$ and $\phi \in K$

$$\langle u_t, \phi\rangle]_0^t = \int_0^t \{\langle u_t, \phi_t\rangle - \langle \mathcal{D}_L u, \mathcal{D}_L\phi\rangle - M(\|Du\|^2)\langle Du, D\phi\rangle - \langle Q(\tau, \cdot, u, u_t), \phi\rangle - \langle f(\cdot, u), \phi\rangle\} \mathrm{d}\tau,$$

and the conservation law (B) stated exactly as that of Section 3. This definition is meaningful under hypotheses (H)$_L$ and (AS)-(a), as justified above. Under the structure assumptions of Theorems 5.4 and 5.5 asymptotic stability carries over equally, that is $Eu(t) = o(1)$ as $t \to \infty$ along any strong solution $u$ of (5.7).

## 6. Applications

Following Sections 5 and 7 of [14], we present here some useful consequences of the main theorems, which can be deduced choosing the auxiliary function $k$ in an appropriate way. Hereby we assume $(\mathrm{H})_L$ and (AS), with $L \geq 1$.

**Corollary 6.1.** *Suppose*

$$\liminf_{t\to\infty} \frac{1}{t}\left\{\left(\int_0^t \varrho \mathrm{d}\tau\right)^{1/2} + \left(\int_0^t \delta_1 \mathrm{d}\tau\right)^{1/m} + \left(\int_0^t \delta_2 \mathrm{d}\tau\right)^{1/q} + \left(\int_0^t \sigma^{1-\wp} \mathrm{d}\tau\right)^{1/\wp}\right\} < \infty, \tag{6.1}$$

*Then the assertions of Theorem 3.1, where $\varrho \equiv 0$, and Theorems 4.1, 5.4 hold.*

*If $\varrho(t) \geq \varrho_0 > 0$ in $\mathbb{R}_0^+$, then the term $\sigma^{1-\wp}$ in (6.1) can be dropped and the assertions of Theorems 4.4, 4.5 and 5.5 hold.*

**Proof.** In the related theorems it is enough to take the auxiliary function $k$ equal to 1. When $\varrho(t) \geq \varrho_0 > 0$, then (4.6) and (4.11) are clearly automatic. □

When the passive viscous damping $Q$ is autonomous, then $\sigma$ and $\delta_1$, $\delta_2$ can be taken in (AS) equal to the same positive constant, and condition (6.1) is automatic since $2 \leq m < q$. The standard case treated in the literature for problems (1.9) and (1.10) is when $Q$ does not depend also on $x$ and $u$, and the coefficient $\varrho$ of the viscous internal material damping is assumed constant. Hence Corollary 6.1 trivially applies in this simplest case. For instance, in the main Theorem 1.2 of [9], a precise estimate for the total energy along the solutions of (1.10) is produced, when, in our notation, $N = 1$, $L = 2$, $a > 0$, that is, in the non-degenerate case, $f \equiv 0$, $\varrho \equiv 0$, $Q = Q(v)$ verifies (AS)-(b) and $(\mathscr{T})$ with $\omega(\tau) = \tau^{m-1}$, $0 \leq \tau \leq 1$, and $m \geq 2$, while (AS)-(a) holds only for $v \geq 1$, with $m = q \in [2, r]$. Hence, Corollary 6.1 extends also to the degenerate case the fact that the asymptotic stability property for (1.10), when (AS)-(a) holds.

The recent paper [17] deals with the vectorial case $N = 2$ of (1.10), when $L = 2$, but $n = 1$, $Q \equiv 0$, $\varrho \equiv 0$, $f = f(u)$ is a special smooth nonlinearity, verifying $(\mathrm{H})_L$ with $p \geq 2$, and again in the non-degenerate case $a > 0$. Hence Corollary 6.1 trivially applies since (6.1) is automatic.

The vectorial case $N = 2$ of (1.9), when $L = 1$, $Q \equiv 0$, $\varrho \equiv 1$, $f = f(u)$ is a special smooth nonlinearity, verifying $(\mathrm{H})_L$ with $2 \leq p \leq 2(n-1)/(n-2) < r$, and again in the non-degenerate case $a > 0$ is treated in [5]. Corollary 6.1 extends also for this model the asymptotic results of [5] to nontrivial dampings $Q$ verifying (AS) and (6.1).

For other similar examples and a more extensive bibliography we refer the reader to [17].

**Corollary 6.2.** *Suppose*

$$\liminf_{t\to\infty} \frac{1}{\log t}\left\{\left(\int_1^t \frac{\varrho}{\tau^2} \mathrm{d}\tau\right)^{1/2} + \left(\int_1^t \frac{\delta_1}{\tau^m} \mathrm{d}\tau\right)^{1/m} + \left(\int_1^t \frac{\delta_2}{\tau^q} \mathrm{d}\tau\right)^{1/q} + \left(\int_1^t \frac{\sigma^{1-\wp}}{\tau^\wp} \mathrm{d}\tau\right)^{1/\wp}\right\} < \infty. \tag{6.2}$$

*Then the assertions of Theorem 3.1, where $\varrho \equiv 0$, and Theorems 4.1, 5.4 hold.*

*If $\varrho(t) \geq \varrho_0/t > 0$ in $\mathbb{R}_0^+$, then the term $\sigma^{1-\wp}/\tau^\wp$ in (6.2) can be dropped and the assertions of Theorems 4.4, 4.5 and 5.5 hold.*

**Proof.** In the related theorems it is enough to take the auxiliary function $k(t) = \min\{1, 1/t\}$. When $\varrho(t) \geq \varrho_0/t > 0$, then (4.6) and (4.11) are clearly automatic. □

**The prototype systems, with f and Q as in (1.7)**

Here we turn back to the systems (1.1) and (1.8)–(1.10), when $f$ and $Q$ are as in (1.7). We first show that $Q$ satisfies (AS). Indeed, letting $h_i(t, x)$ the least eigenvalue of the symmetric part of the continuous coefficient matrix $A_i(t, x)$, with $h_i(t, x) \geq 0$ by assumption, and $H_i(t, x)$ be its Euclidean norm, for $(t, x) \in \mathbb{R}_0^+ \times \Omega$ and $v \in \mathbb{R}^N$ we then have

$$h_i(t, x)|v|^2 \leq (A_i(t, x)v, v), \qquad |A_i(t, x)| \leq H_i(t, x), \quad i = 1, 2.$$

Since $Q(t,x,v) = A_1(t,x)|v|^{m-2}v + A_2(t,x)|v|^{q-2}v$ and $2 \le m < q \le s$ in (1.7), it follows that (AS) is satisfied with

$$\sigma(t) = \inf_{x\in\Omega}\{h_1(t,x) + h_2(t,x)\}, \quad \omega(\tau) = \min\{\tau^q, \tau^2\},$$

provided that $\sigma^{1-\wp} \in L^1_{\text{loc}}(\mathbb{R}^+_0)$, for some $\wp > 1$, and

$$H_i \le \gamma_i h_i \quad \text{in } \mathbb{R}^+_0 \times \Omega \text{ for some } \gamma_i \ge 1, \tag{6.3}$$

and $d_1(t,x) = \gamma_1^{m-1} H_1(t,x)$, $d_2(t,x) = \gamma_2^{q-1} H_2(t,x)$, with $H_1(t\cdot) \in L^{s/(s-m)}(\Omega)$ and $H_2(t\cdot) \in L^{s/(s-q)}(\Omega)$ if $q < s$, or $H_2(t\cdot) \in L^\infty(\Omega)$ if $q = s$. Indeed,

$$\begin{aligned}|Q(t,x,v)| &\le H_1(t,x)|v|^{m-1} + H_2(t,x)|v|^{q-1}\\ &= H_1(t,x)^{1/m}(H_1(t,x)|v|^m)^{1/m'} + H_2(t,x)^{1/q}(H_2(t,x)|v|^q)^{1/q'}\\ &\le \gamma_1^{1/m'} H_1(t,x)^{1/m}(Q(t,x,v),v)^{1/m'} + \gamma_2^{1/q'} H_2(t,x)^{1/q}(Q(t,x,v),v)^{1/q'}.\end{aligned}$$

Condition (6.3) is the *weak uniform definiteness of* $A_i$. Of course, (6.3) *is automatic when either* (1.7) *is scalar, that is,* $N = 1$, *or* $Q = Q(v)$ *is autonomous and independent of* $x$, *so that* $h_i = \text{constant} > 0$. Condition (H) is valid for $f$ in (1.7), as shown in Section 2. Hence Corollaries 6.1 and 6.2 apply, provided that (6.1) or (6.2) hold, respectively.

Motivated by the example (1.7), we introduce the more specific behavior for the function $\omega$ in (AS)-(b)

$$\omega(\tau) = \min\{\tau^\nu, \tau^2\}, \quad \nu \ge q, \tag{6.4}$$

where we recall that $q \le s = \max\{p, r\}$, when $n \ge 3$. Arguing as in [14] we establish

**Theorem 6.3.** *Let the assumptions of either Theorem 3.1, or of Theorem 4.1, or of Theorem 5.4 hold, with the only exception that* $k \notin L^1(\mathbb{R}^+_0)$ *is now of class* $W^{1,1}_{\text{loc}}(\mathbb{R}^+_0) \cap L^\infty(\mathbb{R}^+_0)$ *and satisfies also*

$$|k'| \le \text{Const.}\, \sigma^\lambda k^{1-\lambda} \quad \textit{a.e. in } \mathbb{R}^+_0, \tag{6.5}$$

*where* $\lambda > 0$ *and*

$$\frac{1}{\lambda} \ge \begin{cases}\max\{s', \nu\}, & \text{if } n \ge 3,\\ \nu, & \text{if } n = 2.\end{cases} \tag{6.6}$$

*Then either* (3.5) *or* (5.6) *is valid along any strong solution of* (1.1), (1.8)–(1.10), *respectively.*

**Proof.** It is enough to treat only the case $n \ge 3$. Since $\nu$ in (6.4) can be taken as large as we wish, we may assume, without loss of generality, that $\lambda = 1/\nu$. The proof can follow that of the respective theorems, except for the estimation of the first term $\int_T^t k'\langle u, u_t\rangle d\tau$. For any $\tau \in \mathbb{R}^+_0$ we introduce the sets

$$\Omega_1 = \Omega_1(\tau) = \{x \in \Omega : |u_t(\tau,x)| \le 1\}, \qquad \Omega_2 = \Omega_2(\tau) = \{x \in \Omega : |u_t(\tau,x)| > 1\}.$$

By (6.5), Hölder's inequality, (AS)-(b) and (6.4) we have

$$\begin{aligned}\int_{\Omega_1} k'(u,u_t)dx &\le \text{Const.} \int_{\Omega_1} \sigma^\lambda k^{1-\lambda}|u|\cdot|u_t|dx\\ &\le \text{Const.}\, |\Omega|^{1/\nu'-1/s} k^{1/\nu'}\|u\|_s \left(\int_{\Omega_1}\sigma|u_t|^\nu dx\right)^{1/\nu}\\ &\le \text{Const.}\, k^{1/\nu'}|\langle Q(\tau,\cdot,u,u_t),u_t\rangle|^{1/\nu},\end{aligned}$$

since $\nu' \le s$ and $\|u\|_s \in L^\infty(\mathbb{R}^+_0)$ by (3.7), being $s = \max\{p, r\}$. Now, since $s' < 2$ and $|u_t(\tau,x)| > 1$ in $\Omega_2$, by Hölder's inequality, (AS)-(b) and (3.7),

$$\int_{\Omega_2} k'(u,u_t)dx \le \text{Const.} \int_{\Omega_2} \sigma^\lambda k^{1-\lambda}|u|\cdot|u_t|^{2/s'}dx$$

$$\le \text{Const. } k^{1/\nu'}\|u\|_s \cdot \|u_t\|^{2/\nu'-2/s}\left(\int_{\Omega_2}\sigma|u_t|^2\mathrm{d}x\right)^{1/\nu}$$
$$\le \text{Const. } k^{1/\nu'}|\langle Q(\tau,\cdot,u,u_t),u_t\rangle|^{1/\nu}.$$

Consequently, integrating from $T$ to $t \ge T$, we find

$$\int_T^t k'\langle u,u_t\rangle \mathrm{d}\tau \le \text{Const. } \left(1+\int_T^t k\mathrm{d}\tau\right)\left(\int_T^t |\langle Q(\tau,\cdot,u,u_t),u_t\rangle|\mathrm{d}\tau\right)^{1/\nu}$$

by Hölder's and Young's inequalities. The proof proceeds exactly as in the respective proofs of Theorems 3.1, 4.1 and 5.4. Indeed, in the first two cases, by (3.7), we can take $T$ even large so that,

$$\int_T^t k'\langle u,u_t\rangle \mathrm{d}\tau \le 1+\frac{\alpha}{4}\int_T^t k\mathrm{d}\tau,$$

where $\alpha = \alpha(l)$ is given in Lemma 3.3, while the same estimate is used in the third case $\alpha$ here denotes $\alpha_i = \alpha_i(l_i)$, where $\alpha_i$ are given in Lemma 5.3, $i = 1, 2$. Hence, instead of (3.24), we reach the main formula

$$V(t_i) \le S(T) - \frac{\alpha}{4}\int_T^{t_i} k\mathrm{d}\tau,$$

where $S(T) = 1 + V(T) + \varepsilon(T)\ell[2C(\theta)+1]\int_0^T k\mathrm{d}\tau$, while $\ell$ is given in either (3.22) or (4.5), and $\varepsilon(T) = \max\{\varepsilon_1(T), \varepsilon_2(T), \varepsilon_3(T), \varepsilon_6(T)\}$. □

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