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Global Nonexistence for Nonlinear Kirchhoff Systems

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Abstract

In this paper we consider the problem of non-continuation of solutions of dissipative nonlinear Kirchhoff systems, involving the $p(x)$-Laplacian operator and governed by nonlinear driving forces $f = f(t, x, u)$, as well as nonlinear external damping terms $Q = Q(t, x, u, u_t)$, both of which could significantly dependent on the time t. The theorems are obtained through the study of the natural energy Eu associated to the solutions u of the systems. Thanks to a new approach of the classical potential well and concavity methods, we show the nonexistence of global solutions, when the initial energy is controlled above by a critical value; that is, when the initial data belong to a specific region in the phase plane. Several consequences, interesting in applications, are given in particular subcases. The results are original also for the scalar standard wave equation when $p \equiv 2$ and even for problems linearly damped.

1. Introduction

In this paper we investigate the question of global nonexistence of solutions for dissipative anisotropic nonhomogeneous $p(x)$-Kirchhoff systems. As far as we know, this is the first non-continuation paper concerning with them; for related asymptotic stability problems we refer to [3]. The natural setting is the variable exponent Sobolev space $W_0^{1,p(\cdot)}(\Omega)$, where p varies continuously in Ω. In details, we consider in $\mathbb{R}_0^+ \times \Omega$

$$\begin{cases} u_{tt} - M\left(\mathscr{I}u(t)\right)\Delta_{p(x)}u + \mu|u|^{p(x)-2}u + Q(t,x,u,u_t) = f(t,x,u), \\ u(t,x) = 0 \qquad\qquad \text{on } \mathbb{R}_0^+ \times \partial\Omega, \end{cases} \tag{1.1}$$

where $u = (u_1, \ldots, u_N) = u(t, x)$ is the vectorial displacement, $N \geqq 1$, $\mathbb{R}_0^+ = [0, \infty)$, Ω is a bounded domain of $\mathbb{R}^n$ and $\mu \geqq 0$. Here $\Delta_{p(x)}$ denotes the vectorial

$p(x)$-Laplacian operator defined as $\operatorname{div}(|Du|^{p(x)-2}Du)$, where div is the vectorial divergence and Du the Jacobian matrix of u. While the associated *$p(x)$-Dirichlet energy integral* is $\mathscr{I}u(t) = \int_\Omega \{|Du(t,x)|^{p(x)}/p(x)\}\,dx$. The functions f, M and Q represent a *source force*, a *Kirchhoff dissipative term* and an *external damping term*, respectively.

Throughout the paper, we assume that $(Q(t,x,u,v),v) \geqq 0$ *for all* (t,x,u,v) in $\mathbb{R}_0^+ \times \Omega \times \mathbb{R}^N \times \mathbb{R}^N$,

$$
\begin{aligned}
&Q \in C(\mathbb{R}_0^+ \times \Omega \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N) \quad \text{and} \quad f \in C(\mathbb{R}_0^+ \times \Omega \times \mathbb{R}^N \to \mathbb{R}^N),\\
&f(t,x,u) = F_u(t,x,u), \quad F(t,x,0) = 0,
\end{aligned}
$$

so that $F(t,x,u) = \int_0^1 (f(t,x,\tau u),u)\,d\tau$ is a potential for f in u; and also $(\mathscr{M})$ *the function* $M \in C(\mathbb{R}_0^+ \to \mathbb{R}_0^+)$ *is locally Lipschitz in* $\mathbb{R}^+$ *and such that*

$$
\gamma \mathscr{M}(\tau) \geqq \tau M(\tau), \quad \tau \in \mathbb{R}_0^+, \qquad \mathscr{M}(\tau) = \int_0^\tau M(z)\,dz,
$$

for some number $\gamma \geqq 1$.

The question of non-continuation of solutions is treated by means of the natural energy Eu associated with any solution u of the system, refer to (3.1). The main result is Theorem 3.1, where the initial energy $Eu(0)$ is bounded from above by the critical value $\tilde{E}_1$; see Fig. 1. Refining an argument introduced by PUCCI and SERRIN [22, Theorem 1–(ii)] for evolution systems with linear damping terms, together with a new combination of the classical potential well and concavity methods, in the way used, for example, in [17,18,23] for the wave case, and in [25] for the standard wave Kirchhoff equation, we completely extend the region of global nonexistence for the anisotropic Kirchhoff systems from Σ to $\tilde{\Sigma}$. Indeed, global nonexistence results are proved assuming only that $Eu(0) < \tilde{E}_1$, independently of the initial value $\|u(0,\cdot)\|_{q(\cdot)}$; as an illustrative example we refer to Corollary 4.1.

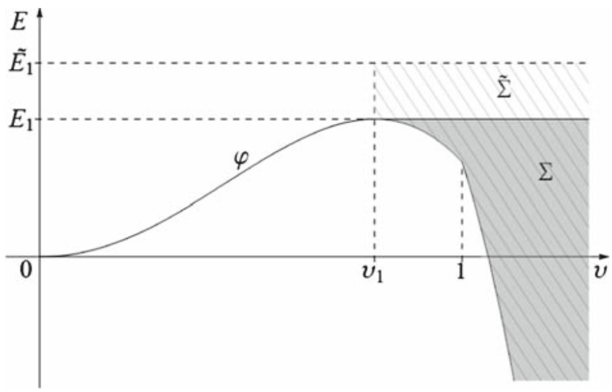

Fig. 1. *The phase plane* (v, E). Here $E = Eu(0)$, where Eu represents the natural energy associated with the solution u of the system, while v stands for $\|u(0,\cdot)\|_{q(\cdot)}$. The global nonexistence results, given in [17,18,22–25] in special subcases of this paper, concern only the region Σ which is smaller than $\tilde{\Sigma}$, when global nonexistence occurs

Here $\|\cdot\|_{q(\cdot)}$ denotes the norm in the anisotropic Lebesgue space $L^{q(\cdot)}(\Omega)$, where $q(\cdot)$ is a variable exponent related to the growth of f in the u variable, see condition $(\mathscr{F}_3)$ in Section 3, or the application given in the example (4.1)–(4.2). The results of this paper are new even in the standard wave case when $p \equiv 2$ as well as when Q is linear in the $v = u_t$ variable.

Several consequences are deduced in special subcases of f, M and Q, interesting in applications; see Section 4. We also show that *if* u *is a global solution of* (1.1) *and* $Eu(0) \leqq E_1$, *then* $\tilde{E}_1 \leqq E_1$. This result allows us to prove in a new way and wider setting that *if* u *is a local solution of* (1.1), *with* $\|u(0,\cdot)\|_{q(\cdot)} > v_1$ *and* $Eu(0) = E_1$, *then* u *cannot be global*, see Theorem 4.3. The possibility to cover this case was first discovered by Vitillaro [23], but with a different proof technique and only for *strong* solutions, even if not explicitly stated, refer to [23, the proof of case (a) of Theorem 3]. When f significantly depends on t, Theorem 4.3 handles also the case $Q \equiv 0$, not covered in [23], refer to Remark 4.1. Indeed, we are able to establish refined results, even dealing with a wider class of solutions verifying a *weak conservation law*, thanks to a new argument based on the lower bounds of the potential energy along a solution u. The main reason to consider weak solutions was first given in [20, Remark 4 at page 199]; see also [21, Remark 2 at page 49] and the discussion on [17, page 345].

In recent years, the study of p-Kirchhoff equations involving the quasilinear homogeneous p-Laplace operator, based on the theory of standard Sobolev spaces $W_0^{1,p}(\Omega)$, has been widely studied, see, for example, [6,25], while for wave equations [7], and for the elliptic case [5,8]. In particular, in [25] global nonexistence results are proved for scalar Kirchhoff equations, when $Eu(0) < E_1$ and all the exponents are constant, with $p(x) \equiv 2$. In addition, in [25] the conservation law is assumed only in the stronger form $(B)_s$, so that the energy function Eu is nonincreasing in $\mathbb{R}_0^+$ (see Sections 3 and 4 for the definition of $(B)_s$ and Remark 5.1). Finally, in [25] the damping function Q depends only on v, while f is a pure power of u. Here we cover much more situations, in which the dependence on t could be significant.

Last but not least, we recall that the nonhomogeneous $p(x)$-Kirchhoff operator has been used in the last decades to model various phenomena, see [9–16,19,26,27] and the references therein. Indeed, recently, there has been an increasing interest in studying systems involving somehow nonhomogeneous $p(x)$–Laplace operators, motivated by the image restoration problem, by the modeling of electro-rheological fluids (sometimes referred to as *smart fluids*), as well as the thermo-convective flows of non-Newtonian fluids: details and further references can be found in [2 and 19], while for the regularity of weak solutions we refer to [1].

2. Basic facts and notation

Let $h \in C_+(\overline{\Omega})$, where

$$C_+(\overline{\Omega}) = \{h \in C(\overline{\Omega}) : \min_{x\in\overline{\Omega}} h(x) > 1\},$$

and define

$$h_+ = \sup_{x\in\Omega} h(x) \quad \text{and} \quad h_- = \inf_{x\in\Omega} h(x).$$

Hereafter $p \in C_+(\overline{\Omega})$ is fixed. The *variable exponent Lebesgue space*, denoted by $L^{p(\cdot)}(\Omega) = [L^{p(\cdot)}(\Omega)]^N$ and consisting of all the measurable vector-valued functions $u : \Omega \to \mathbb{R}^N$ such that $\int_\Omega |u(x)|^{p(x)}\,dx$ is finite, is endowed with the so called *Luxemburg norm*

$$\|u\|_{p(\cdot)} = \inf\left\{\lambda > 0 : \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)} dx \leqq 1\right\},$$

and is a separable and reflexive Banach space, refer to [16, Corollaries 2.12 and 2.7]. For basic properties of the variable exponent Lebesgue spaces we refer to [16]. Since here $0 < |\Omega| < \infty$, if $q \in C_+(\overline{\Omega})$ and $p \leqq q$ in Ω, then the embedding $L^{q(\cdot)}(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega)$ is continuous, refer to [16, Theorem 2.8].

Let $L^{p'(\cdot)}(\Omega)$ be the conjugate space of $L^{p(\cdot)}(\Omega)$, obtained by conjugating the exponent pointwise that is, $1/p(x) + 1/p'(x) = 1$, [10, Theorem 1.14]. For any $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$ the following Hölder type inequality

$$\left|\int_\Omega (u(x), v(x))\,dx\right| \leqq \left(\frac{1}{p_-} + \frac{1}{p'_-}\right) \|u\|_{p(\cdot)} \|v\|_{p'(\cdot)}$$

is valid, where $(\cdot,\cdot)$ is the inner product on $\mathbb{R}^N \times \mathbb{R}^N$, [16, Theorem 2.1].

An important role in manipulating the generalized Lebesgue–Sobolev spaces is played by the $p(\cdot)$-*modular* of the $L^{p(\cdot)}(\Omega)$ space, which is the convex function $\rho_{p(\cdot)} : L^{p(\cdot)}(\Omega) \to \mathbb{R}$ defined by

$$\rho_{p(\cdot)}(u) = \int_\Omega |u(x)|^{p(x)}\,dx.$$

If $(u_j)_j, u \in L^{p(\cdot)}(\Omega)$, then the following relations hold: $\|u\|_{p(\cdot)} < 1$ $(= 1;\ > 1) \Leftrightarrow \rho_{p(\cdot)}(u) < 1$ $(= 1;\ > 1)$,

$$\begin{aligned} \|u\|_{p(\cdot)} \geqq 1 &\Rightarrow \|u\|_{p(\cdot)}^{p_-} \leqq \rho_{p(\cdot)}(u) \leqq \|u\|_{p(\cdot)}^{p_+}, \\ \|u\|_{p(\cdot)} \leqq 1 &\Rightarrow \|u\|_{p(\cdot)}^{p_+} \leqq \rho_{p(\cdot)}(u) \leqq \|u\|_{p(\cdot)}^{p_-}, \end{aligned} \tag{2.1}$$

and $\|u_j - u\|_{p(\cdot)} \to 0 \Leftrightarrow \rho_{p(\cdot)}(u_j - u) \to 0 \Leftrightarrow (u_j)_j$ converges to u in measure in Ω and $\rho_{p(\cdot)}(u_j) \to \rho_{p(\cdot)}(u)$, since $p_+ < \infty$. In particular, $\rho_{p(\cdot)}$ is continuous in $L^{p(\cdot)}(\Omega)$. For a proof of these facts see [10, Theorem 1.4] and [16].

The *variable exponent Sobolev space* $W^{1,p(\cdot)}(\Omega) = [W^{1,p(\cdot)}(\Omega)]^N$, consisting of functions $u \in L^{p(\cdot)}(\Omega)$ whose distributional Jacobian matrix Du is in $[L^{p(\cdot)}(\Omega)]^{nN}$, is endowed with the norm

$$\|u\|_{1,p(\cdot)} = \|u\|_{p(\cdot)} + \|Du\|_{p(\cdot)}.$$

Thus $W^{1,p(\cdot)}(\Omega)$ is a separable and reflexive Banach space, refer to [16, Theorem 3.1]. Define $H_0^{1,p(\cdot)}(\Omega) = [H_0^{1,p(\cdot)}(\Omega)]^N$ as the closure of $C_0^\infty(\Omega) = [C_0^\infty(\Omega)]^N$

in $W^{1,p(\cdot)}(\Omega)$, and $W_0^{1,p(\cdot)}(\Omega) = [W_0^{1,p(\cdot)}(\Omega)]^N$ as the Sobolev space of the functions $u \in W^{1,p(\cdot)}(\Omega)$, *with zero boundary values*, refer to [12]. As shown by Zhikov [26,27], the smooth functions are in general not dense in $W^{1,p(\cdot)}(\Omega)$, but if $p \in C_+(\overline{\Omega})$ is *logarithmic Hölder continuous*, that is, there exists $L > 0$ such that for all $x, y \in \Omega$, with $0 < |x - y| \leqq 1/2$

$$|p(x) - p(y)| \leqq -\frac{L}{\log(|x-y|)}, \tag{2.2}$$

then $H_0^{1,p(\cdot)}(\Omega) = W_0^{1,p(\cdot)}(\Omega)$, namely the density property holds, see [12,15] and in particular [14, Theorem 3.3]. Since Ω is a bounded domain, if $p \in C_+(\overline{\Omega})$ satisfies (2.2), then the $p(\cdot)$-*Poincaré inequality*

$$\|u\|_{p(\cdot)} \leqq C\|Du\|_{p(\cdot)}$$

is valid for all $u \in W_0^{1,p(\cdot)}(\Omega)$, where C depends on p, $|\Omega|$, $\operatorname{diam}(\Omega)$, n and N, [14, Theorem 4.1], and so

$$\|u\| = \|Du\|_{p(\cdot)},$$

is an equivalent norm in $W_0^{1,p(\cdot)}(\Omega)$. Moreover $W_0^{1,p(\cdot)}(\Omega)$ is a separable and reflexive Banach space. If $p_+ < n$ and (2.2) holds, then the embedding $W_0^{1,p(\cdot)}(\Omega) \hookrightarrow L^{p^*(\cdot)}(\Omega)$ is continuous, see [13, Theorem 1.1], where p^* *is the critical variable exponent related to* p, defined by the relation

$$p^*(x) = \frac{np(x)}{n - p(x)} \quad \text{for all} \quad x \in \Omega.$$

Details, extensions and further references can be found in [10,12–16].

Hereafter, we assume that

$$p \in C_+(\overline{\Omega}) \text{ satisfies (2.2) and } 1 < p_- \leqq p_+ < n.$$

For all $h \in C(\overline{\Omega})$, with $1 \leqq h \leqq p^*$ in Ω, we denote by $\lambda_{h(\cdot)}$ the Sobolev constant, depending on h, p, $|\Omega|$, n and N, of the continuous embedding $W_0^{1,p(\cdot)}(\Omega) \hookrightarrow L^{h(\cdot)}(\Omega)$, that is

$$\|u\|_{h(\cdot)} \leqq \lambda_{h(\cdot)}\|Du\|_{p(\cdot)} \quad \text{for all} \quad u \in W_0^{1,p(\cdot)}(\Omega), \tag{2.3}$$

see [13, Theorem 1.1] and also [16, Theorem 2.8]. Of course in (2.3) we can take $h \equiv 1$.

For simplicity in notation

$$L^{p(\cdot)}(\Omega) = [L^{p(\cdot)}(\Omega)]^N, \quad W_0^{1,p(\cdot)}(\Omega) = [W_0^{1,p(\cdot)}(\Omega)]^N,$$

endowed with the norms $\|\cdot\|_{p(\cdot)}$ and $\|u\| = \|Du\|_{p(\cdot)}$, respectively.

Throughout the paper, the usual Lebesgue space $L^2(\Omega) = [L^2(\Omega)]^N$ is equipped with the canonical norm $\|\varphi\|_2 = \left(\int_\Omega |\varphi(x)|^2\,dx\right)^{1/2}$, while the elementary bracket pairing $\langle\varphi, \psi\rangle = \int_\Omega (\varphi(x), \psi(x))\,dx$ is clearly well defined for all φ, ψ such that $(\varphi, \psi) \in L^1(\Omega)$. Finally

$$K = C(\mathbb{R}_0^+ \to W_0^{1,p(\cdot)}(\Omega)) \cap C^1(\mathbb{R}_0^+ \to L^2(\Omega))$$

denotes *the main solution and test function space*, adopted in Sections 3 and 4.

3. The main theorem

In this section we provide a non-continuation result for the solutions of the problem (1.1) and assume that f, M and Q are as in the Section 1. Suppose also that *for all* $\phi \in K$

$$(\mathscr{F}_1)\quad \begin{gathered} F(t,\cdot,\phi(t,\cdot)),\ (f(t,\cdot,\phi(t,\cdot)),\phi(t,\cdot)) \in L^1(\Omega) \quad for\ all \quad t \in \mathbb{R}_0^+; \\ \langle f(t,\cdot,\phi(t,\cdot)),\phi(t,\cdot)\rangle \in L^1_{\mathrm{loc}}(\mathbb{R}_0^+). \end{gathered}$$

The *potential energy of the field* $\phi \in K$ is given by

$$\mathscr{F}\phi(t) = \mathscr{F}(t,\phi) = \int_\Omega F(t,x,\phi(t,x))\,\mathrm{d}x,$$

and it is well defined by $(\mathscr{F}_1)$, while the natural *total energy of the field* $\phi \in K$, associated with the problem (1.1), is

$$\begin{aligned} E\phi(t) &= \tfrac{1}{2}\|\phi_t(t,\cdot)\|_2^2 + \mathscr{A}\phi(t) - \mathscr{F}\phi(t), \\ \mathscr{A}\phi(t) &= \mathscr{M}(\mathscr{I}\phi(t)) + \mu\int_\Omega \frac{|\phi(t,x)|^{p(x)}}{p(x)}\,\mathrm{d}x \geqq 0, \end{aligned} \tag{3.1}$$

where $\mu \geqq 0$ and $\mathscr{I}\phi$ is the $p(\cdot)$-*Dirichlet energy integral*, that is

$$\mathscr{I}\phi(t) = \mathscr{I}(t,\phi) = \int_\Omega \frac{|D\phi(t,x)|^{p(x)}}{p(x)}\,\mathrm{d}x.$$

Of course $E\phi$ is well defined in K. For all $\phi \in K$ and $(t,x) \in \mathbb{R}_0^+ \times \Omega$ we define pointwise

$$A\phi(t,x) = -M(\mathscr{I}\phi(t))\Delta_{p(x)}\phi(t,x) + \mu|\phi(t,x)|^{p(x)-2}\phi(t,x), \tag{3.2}$$

so that A is the Fréchet derivative of $\mathscr{A}$ with respect to ϕ. By $(\mathscr{M})$ we have

$$\begin{aligned} \langle A\phi(t,\cdot),\phi(t,\cdot)\rangle &= M(\mathscr{I}\phi(t))\rho_{p(\cdot)}(D\phi(t,\cdot)) + \mu\rho_{p(\cdot)}(\phi(t,\cdot)) \\ &\leqq p_+\left\{\mathscr{I}\phi(t)M(\mathscr{I}\phi(t)) + \mu\int_\Omega \frac{|\phi(t,x)|^{p(x)}}{p(x)}\,\mathrm{d}x\right\} \\ &\leqq \gamma p_+\mathscr{A}\phi(t). \end{aligned} \tag{3.3}$$

Before introducing the definition of the solution to (1.1), we assume the following monotonicity condition

$$(\mathscr{F}_2)\qquad \mathscr{F}_t \geqq 0 \quad in \quad \mathbb{R}_0^+ \times W_0^{1,p(\cdot)}(\Omega),$$

where $\mathscr{F}_t$ is the partial derivative with respect to t of $\mathscr{F} = \mathscr{F}(t,w)$, with $(t,w) \in \mathbb{R}_0^+ \times W_0^{1,p(\cdot)}(\Omega)$.

Following [3 and 20], we say that u is a (*weak*) *solution of* (1.1) if $u \in K$ satisfies the two properties:

(A) *Distribution Identity*

$$\langle u_t, \phi\rangle\,\big]_0^t = \int_0^t \{\, \langle u_t, \phi_t\rangle - M(\mathscr{I}u(\tau))\cdot\langle |Du|^{p(\cdot)-2}Du, D\phi\rangle \\ -\mu\langle |u|^{p(\cdot)-2}u, \phi\rangle - \langle Q(\tau,\cdot,u,u_t) - f(\tau,\cdot,u), \phi\rangle\}\,\mathrm{d}\tau$$

for all $t \in \mathbb{R}_0^+$ and $\phi \in K$;

(B) *Energy Conservation*

(i) $\mathscr{D}u(t) = \langle Q(t,\cdot,u(t,\cdot),u_t(t,\cdot)), u_t(t,\cdot)\rangle + \mathscr{F}_t u(t) \in L^1_{\mathrm{loc}}(\mathbb{R}_0^+)$,

(ii) $Eu(t) \leqq Eu(0) - \int_0^t \mathscr{D}u(\tau)\,\mathrm{d}\tau$ for all $t \in \mathbb{R}_0^+$.

In general it is important to consider (weak) solutions instead of strong solutions, namely functions $u \in K$ satisfying (A), (B)-(i), while (B)-(ii) is replaced by the *Strong Energy Conservation* $(\mathrm{B})_{\mathrm{s}}$-(ii), that is $Eu(t) = Eu(0) - \int_0^t \mathscr{D}u(\tau)\,\mathrm{d}\tau$ for all $t \in \mathbb{R}_0^+$. The main reason was first given in [20, Remark 4 at page 199]; see also [21, Remark 2 at page 49] and the discussion in [17, page 345]. Of course if u is a strong solution, then Eu *is non-increasing in* $\mathbb{R}_0^+$ and this makes the analysis much simpler. We refer also to the Remark 4.1.

Remark 3.1. If $u \in K$ is a solution of (1.1) in $\mathbb{R}_0^+ \times \Omega$, then by $(3.1)_2$ there exists always $w_1 \geqq 0$ such that $\mathscr{A}u(t) \geqq w_1$ for all $t \in \mathbb{R}_0^+$. Hence by $(3.1)_1$, (B)-(ii) and $(\mathscr{F}_2)$ we get $\mathscr{F}u(t) \geqq w_1 - Eu(0) \geqq -Eu(0)$ for all $t \in \mathbb{R}_0^+$, in other words $\mathscr{F}u$ *is bounded below in* $\mathbb{R}_0^+$ *along any solution* $u \in K$.

In order to state our main result we consider the following condition:
$(\mathscr{F}_3)$ *There exists a function* $q \in C_+(\overline{\Omega})$ *satisfying the restriction*

$$\max\{2, \gamma p_+\} < q_-, \tag{3.4}$$

with the property that for all $\mathfrak{F} > 0$ *and* $\phi \in K$ *for which* $\inf_{t\in\mathbb{R}_0^+}\mathscr{F}\phi(t) \geqq \mathfrak{F}$, *there exist* $c_1 = c_1(\mathfrak{F},\phi) > 0$ *and* $\varepsilon_0 = \varepsilon_0(\mathfrak{F},\phi) > 0$, *such that*

(i)

$$\mathscr{F}\phi(t) \leqq c_1\rho_{q(\cdot)}(\phi(t,\cdot)) \quad \textit{for all} \quad t \in \mathbb{R}_0^+,$$

and for all $\varepsilon \in (0, \varepsilon_0)$ *there exists* $c_2 = c_2(\mathfrak{F},\phi,\varepsilon) > 0$, *such that*

(ii)

$$\langle f(t,\cdot,\phi(t,\cdot)), \phi(t,\cdot)\rangle - (q_- - \varepsilon)\mathscr{F}\phi(t) \geqq c_2\rho_{q(\cdot)}(\phi(t,\cdot)) \quad \textit{for all} \quad t \in \mathbb{R}_0^+.$$

In general the function $q \in C_+(\overline{\Omega})$ verifies further restrictions than (3.4) in order to get the validity of $(\mathscr{F}_1)$ and $(\mathscr{F}_2)$, see Section 4 for concrete examples.

Theorem 3.1. *Assume* $(\mathscr{M})$, $(\mathscr{F}_1)$ *and* $(\mathscr{F}_2)$. *If* $u \in K$ *is a solution of* (1.1) *in* $\mathbb{R}_0^+ \times \Omega$, *then* $w_2 = \inf_{t\in\mathbb{R}_0^+}\mathscr{F}u(t) > -\infty$. *If there exists* $\varpi > -1$ *such that* $Eu(0) < \varpi w_2$, *then* $w_2 > 0$.

Finally, if also $(\mathscr{F}_3)$ *holds, then there are no solutions* $u \in K$ *of* (1.1) *in* $\mathbb{R}_0^+ \times \Omega$, *for which*

$$Eu(0) < \left(\frac{q_-}{\gamma p_+} - 1\right) w_2 = \tilde{E}_1, \tag{3.5}$$

and for which there exist $T \geqq 0$, $q_1 > 0$, $m > 1$, $\kappa \geqq -m$, *with* $m + \kappa < q_-$, *and non-negative functions* $\delta \in L^\infty_{\mathrm{loc}}(J)$, $\psi, k \in W^{1,1}_{\mathrm{loc}}(J)$, $J = [T, \infty)$, *with* $k' \geqq 0$, $\psi > 0$ *in* J *and* $\psi'(t) = o(\psi(t))$ *as* $t \to \infty$, *such that*

$(\mathscr{Q})$ $\langle Q(t,\cdot,u(t,\cdot),u_t(t,\cdot)), u(t,\cdot)\rangle \leqq q_1 \delta(t)^{1/m} \mathscr{D}u(t)^{1/m'} \|u(t,\cdot)\|_{q(\cdot)}^{1+\kappa/m}$

for all $t \in J$, *and*

$$\delta \leqq (k/\psi)^{m-1} \ \text{ in } J, \qquad \int^\infty \psi(t)\,[\max\{k(t),\psi(t)\}]^{-(1+\theta)}\,\mathrm{d}t = \infty, \tag{3.6}$$

for some appropriate constant $\theta \in (0, \theta_0)$, *where*

$$\theta_0 = \min\left\{\frac{q_- - 2}{q_- + 2},\ \frac{q_- - m - \kappa}{m(1+q_+) + \kappa - q_-}\right\}. \tag{3.7}$$

Proof. Let $u \in K$ be a solution of (1.1) in $\mathbb{R}_0^+ \times \Omega$. Clearly $\mathscr{A}u$ and $\mathscr{F}u$ are bounded below in $\mathbb{R}_0^+$ as shown in Remark 3.1. In particular $w_2 > -\infty$ and $\inf_{t\in\mathbb{R}_0^+} \mathscr{A}u(t) \geqq w_1$ for some $w_1 \geqq 0$. Assume that $Eu(0) < \varpi w_2$, with $\varpi > -1$. Then $\mathscr{F}u(t) \geqq w_1 - Eu(0) > w_1 - \varpi w_2$, which gives $w_2 > w_1/(1+\varpi) \geqq 0$, and so $w_2 > 0$.

Suppose now that also $(\mathscr{F}_3)$ holds and by contradiction that there exists a solution $u \in K$ of (1.1) in $\mathbb{R}_0^+ \times \Omega$, satisfying $(\mathscr{Q})$ and (3.5)–(3.7) as in the statement. Then $\tilde{E}_1 > 0$ since (3.4) is in charge, so that $\varpi = -1 + q_-/\gamma p_+ > -1$. By the first part of the theorem $w_2 > 0$ and so $\tilde{E}_1 > 0$ by (3.4). Fix $E_2 \geqq 0$ in the interval $(Eu(0), \tilde{E}_1)$ and define the function

$$\mathscr{H}(t) = E_2 - Eu(0) + \int_0^t \mathscr{D}u(\tau)\,\mathrm{d}\tau$$

for each $t \in \mathbb{R}_0^+$. Of course $\mathscr{H}$ is well defined and non-decreasing by (B)-(i) and $(\mathscr{F}_2)$, being $\mathscr{D} \geqq 0$ and finite along u. Hence, by (B)-(ii),

$$E_2 - Eu(t) \geqq \mathscr{H}(t) \geqq \mathscr{H}_0 = E_2 - Eu(0) > 0 \quad \text{for} \quad t \in \mathbb{R}_0^+, \tag{3.8}$$

where $\mathscr{H}_0 = \mathscr{H}(0)$. Moreover, by (3.8), (3.1), the choice of E_2 and the definition of w_2, it follows that

$$\begin{aligned}\mathscr{H}(t) &\leqq E_2 - Eu(t) < \tilde{E}_1 + \mathscr{F}u(t) \leqq \left(\frac{q_-}{\gamma p_+} - 1\right)\mathscr{F}u(t) + \mathscr{F}u(t)\\ &= \frac{q_-}{\gamma p_+}\mathscr{F}u(t)\end{aligned} \tag{3.9}$$

for all $t \in \mathbb{R}_0^+$. In correspondence to $\mathfrak{F} = w_2 > 0$, $\phi = u \in K$, there exists $\varepsilon_0 = \varepsilon_0(w_2, u) > 0$ such that ($\mathscr{F}_3$) holds true and, without loss of generality, we take $\varepsilon_0 > 0$ so small that

$$\varepsilon_0 w_2 \leqq (q_- - \gamma p_+)w_2 - \gamma p_+ E_2, \tag{3.10}$$

which is possible since $w_2 > 0$ and $E_2 < \tilde{E}_1$. Note that (3.10) forces $\varepsilon_0 \leqq q_- - \gamma p_+$, being $E_2 \geqq 0$.

Fix $\varepsilon \in (0, \varepsilon_0)$ and take $\phi = u$ in the *Distribution Identity* (A). By (3.1)

$$\begin{aligned}\frac{d}{dt}\langle u_t(t,\cdot), u(t,\cdot)\rangle = c_3\|u_t(t,\cdot)\|_2^2 &+ (q_- - \varepsilon)\mathscr{A}u(t) - \langle Au(t,\cdot), u(t,\cdot)\rangle \\ &+ \langle f(t,\cdot,u(t,\cdot)), u(t,\cdot)\rangle - (q_- - \varepsilon)\mathscr{F}u(t) \\ &- (q_- - \varepsilon)Eu(t) - \langle Q(t,\cdot,u(t,\cdot),u_t(t,\cdot)), u(t,\cdot)\rangle,\end{aligned}$$

where $c_3 = 1 + (q_- - \varepsilon)/2 > 0$ by the choice of ε. By (3.3), applying ($\mathscr{F}_3$)-(ii) with $c_2 = c_2(w_2, u, \varepsilon) > 0$, we obtain for all $t \in \mathbb{R}_0^+$

$$\begin{aligned}\frac{d}{dt}\langle u_t(t,\cdot), u(t,\cdot)\rangle \geqq c_3\|u_t(t,\cdot)\|_2^2 &+ c_2\rho_{q(\cdot)}(u(t,\cdot)) - (q_- - \varepsilon)Eu(t) \\ &- \langle Q(t,\cdot,u(t,\cdot),u_t(t,\cdot)), u(t,\cdot)\rangle + (q_- - \varepsilon - \gamma p_+)\mathscr{A}u(t).\end{aligned}$$

Hence by (3.1) and by the definition of w_2, since $\varepsilon < q_- - \gamma p_+$ by (3.10) and $Eu \leqq E_2 - \mathscr{H}$ by (3.8), we have

$$\begin{aligned}\frac{d}{dt}\langle u_t(t,\cdot), u(t,\cdot)\rangle &\geqq \tilde{c}_3\|u_t(t,\cdot)\|_2^2 + c_2\rho_{q(\cdot)}(u(t,\cdot)) + (q_- - \varepsilon - \gamma p_+)w_2 \\ &\quad - \langle Q(t,\cdot,u(t,\cdot),u_t(t,\cdot)), u(t,\cdot)\rangle - \gamma p_+ Eu(t) \\ &\geqq \tilde{c}_3\|u_t(t,\cdot)\|_2^2 + c_2\rho_{q(\cdot)}(u(t,\cdot)) + (q_- - \varepsilon - \gamma p_+)w_2 \\ &\quad - \langle Q(t,\cdot,u(t,\cdot),u_t(t,\cdot)), u(t,\cdot)\rangle + \gamma p_+\mathscr{H}(t) - \gamma p_+ E_2,\end{aligned}$$

where $\tilde{c}_3 = 1 + \gamma p_+/2$. Consequently,

$$\begin{aligned}\frac{d}{dt}\langle u_t(t,\cdot), u(t,\cdot)\rangle \geqq \tilde{c}_3\|u_t(t,\cdot)\|_2^2 &+ c_2\rho_{q(\cdot)}(u(t,\cdot)) + \gamma p_+\mathscr{H}(t) \\ &- \langle Q(t,\cdot,u(t,\cdot),u_t(t,\cdot)), u(t,\cdot)\rangle,\end{aligned} \tag{3.11}$$

again by (3.10).

By ($\mathscr{F}_3$)-(i), if $\|u(t,\cdot)\|_{q(\cdot)} \geqq 1$ then $\mathscr{F}u(t) \leqq c_1\|u(t,\cdot)\|_{q(\cdot)}^{q_+}$ by $(2.1)_1$. On the other hand, if $\|u(t,\cdot)\|_{q(\cdot)} \leqq 1$ then $w_2 \leqq c_1\|u(t,\cdot)\|_{q(\cdot)}^{q_-}$ by ($\mathscr{F}_3$)-(i), the definition of w_2 and $(2.1)_2$. Hence $\|u(t,\cdot)\|_{q(\cdot)} \geqq (w_2/c_1)^{1/q_-} > 0$, so that $\mathscr{F}u(t) \leqq c_1\rho_{q(\cdot)}(u(t,\cdot)) \leqq c_1\,(c_1/w_2)^{q_+/q_-}\,\|u(t,\cdot)\|_{q(\cdot)}^{q_+}$ by ($\mathscr{F}_3$)-(i). In conclusion, along the solution u, we have for all $t \in \mathbb{R}_0^+$

$$\mathscr{F}u(t) \leqq \tilde{c}_1\|u(t,\cdot)\|_{q(\cdot)}^{q_+}, \quad \text{with } \tilde{c}_1 = \max\{c_1, c_1(c_1/w_2)^{q_+/q_-}\}.$$

Hence, by ($\mathcal{Q}$) and (3.9)

$$\begin{aligned}&\langle Q(t,\cdot,u(t,\cdot),u_t(t,\cdot)),u(t,\cdot)\rangle\\&\leqq q_1\|u(t,\cdot)\|_{q(\cdot)}^{q_-/m}\Big(\delta(t)^{1/(m-1)}\mathscr{D}u(t)\Big)^{1/m'}\|u(t,\cdot)\|_{q(\cdot)}^{-q_+\bar{r}}\\&\leqq \tilde{c}_1^{\bar{r}}q_1\|u(t,\cdot)\|_{q(\cdot)}^{q_-/m}\Big(\delta(t)^{1/(m-1)}\mathscr{D}u(t)\Big)^{1/m'}[\mathscr{F}u(t)]^{-\bar{r}}\\&\leqq c_4\|u(t,\cdot)\|_{q(\cdot)}^{q_-/m}\Big(\delta(t)^{1/(m-1)}\mathscr{D}u(t)\Big)^{1/m'}[\mathscr{H}(t)]^{-\bar{r}}\end{aligned}$$

for all $t\in J$, where $c_4=(\tilde{c}_1q_-/\gamma p_+)^{\bar{r}}q_1$ and

$$\bar{r}=\frac{q_--m-\kappa}{mq_+}\in(0,1),\tag{3.12}$$

since $m+\kappa<q_-$ and $\kappa\geqq-m$ by ($\mathcal{Q}$). Put

$$r_0=\min\left\{\bar{r},\frac{1}{2}-\frac{1}{q_-}\right\}.\tag{3.13}$$

Note that θ_0 in (3.7) can be expressed as $\theta_0=r_0/(1-r_0)$, and take from now on $r=\theta/(1+\theta)$, so that $r\in(0,r_0)$. Consequently, by Young's inequality, we get

$$\begin{aligned}&\langle Q(t,\cdot,u(t,\cdot),u_t(t,\cdot)),u(t,\cdot)\rangle\\&\leqq\left\{(c_4\ell)^m\|u(t,\cdot)\|_{q(\cdot)}^{q_-}+\ell^{-m'}\delta(t)^{1/(m-1)}\mathscr{D}u(t)\right\}[\mathscr{H}(t)]^{-\bar{r}}\\&\leqq(c_4\ell)^m\mathscr{H}_0^{-\bar{r}}\|u(t,\cdot)\|_{q(\cdot)}^{q_-}+\ell^{-m'}\mathscr{H}_0^{r-\bar{r}}\delta(t)^{1/(m-1)}[\mathscr{H}(t)]^{-r}\mathscr{D}u(t),\end{aligned}\tag{3.14}$$

where in the last step we have used the facts that $\mathscr{H}\geqq\mathscr{H}_0$ by (3.8) and that $0<r<r_0\leqq\bar{r}$ by (3.13). The parameter $\ell>0$ will be fixed later. Since $\mathscr{D}u=\mathscr{H}'$, we see that $(1-r)\mathscr{H}^{-r}\mathscr{H}'=[\mathscr{H}^{1-r}]'$. Hence it is convenient to introduce the function

$$\mathscr{Z}=\mathscr{Z}(t)=\lambda k(t)\,[\mathscr{H}(t)]^{1-r}+\psi(t)\langle u_t,u\rangle,$$

where $\lambda>0$ is a constant to be fixed later. Clearly $\mathscr{Z}\in W^{1,1}_{\mathrm{loc}}(J)$ by Corollary VIII.9 of [4] and so, almost everywhere in J,

$$\mathscr{Z}'=\lambda k(1-r)\mathscr{H}^{-r}\mathscr{H}'+\lambda k'\mathscr{H}^{1-r}+\psi\frac{d}{dt}\langle u_t,u\rangle+\psi'\langle u_t,u\rangle.$$

If $t\in\mathbb{R}_0^+$ and $\rho_{q(\cdot)}(u(t,\cdot))\geqq1$, then $\rho_{q(\cdot)}(u(t,\cdot))\geqq\|u(t,\cdot)\|_{q(\cdot)}^{q_-}$, by $(2.1)_1$. Otherwise $\rho_{q(\cdot)}(u(t,\cdot))\leqq1$, so that by ($\mathscr{F}_3$)-(i) we have $w_2\leqq c_1$ and $\rho_{q(\cdot)}(u(t,\cdot))\geqq w_2\|u(t,\cdot)\|_{q(\cdot)}^{q_-}/c_1$. Hence for all $t\in\mathbb{R}_0^+$ we get $\rho_{q(\cdot)}(u(t,\cdot))\geqq\min\{1,w_2/c_1\}\,\|u(t,\cdot)\|_{q(\cdot)}^{q_-}$. Therefore, by (3.11) and (3.14), almost everywhere in J

$$\begin{aligned}\mathscr{Z}'&\geqq\lambda k(1-r)\mathscr{H}^{-r}\mathscr{H}'+\lambda k'\mathscr{H}^{1-r}+\psi'\langle u_t,u\rangle\\&\quad+\psi\left\{\tilde{c}_3\|u_t\|_2^2+c_2\rho_{q(\cdot)}(u)+\gamma p_+\mathscr{H}-\langle Q(t,\cdot,u,u_t),u\rangle\right\}\\&\geqq\Big(\lambda k(1-r)-\ell^{-m'}\mathscr{H}_0^{r-\bar{r}}\delta^{1/(m-1)}\psi\Big)\mathscr{H}^{-r}\mathscr{H}'+\gamma p_+\psi\mathscr{H}+\lambda k'\mathscr{H}^{1-r}\\&\quad+\psi'\langle u_t,u\rangle+\psi\left\{\tilde{c}_3\|u_t\|_2^2+\tilde{c}_2\|u\|_{q(\cdot)}^{q_-}-(c_4\ell)^m\mathscr{H}_0^{-\bar{r}}\|u\|_{q(\cdot)}^{q_-}\right\},\end{aligned}$$

where $\tilde{c}_2 = \min\{c_2,\ c_2 w_2/c_1\}$. Thus, almost everywhere in J, by (3.6)$_1$ and the fact that $\lambda k' \mathscr{H}^{1-r} \geqq 0$, we find

$$\mathscr{Z}' \geqq k\left(\lambda(1-r) - \ell^{-m'}\mathscr{H}_0^{r-\bar{r}}\right)\mathscr{H}^{-r}\mathscr{H}' + \gamma p_+ \psi \mathscr{H} + \psi'\langle u_t, u\rangle + \psi\left\{\tilde{c}_3\|u_t\|_2^2 + \tilde{c}_2\|u\|_{q(\cdot)}^{q_-} - (c_4\ell)^m \mathscr{H}_0^{-\bar{r}}\|u\|_{q(\cdot)}^{q_-}\right\}.$$

Next, from Cauchy's and Young's inequalities, and the definition of K, we have

$$|\langle u_t(t,\cdot), u(t,\cdot)\rangle| \leqq \|u_t(t,\cdot)\|_2\|u(t,\cdot)\|_2 \leqq \|u_t(t,\cdot)\|_2^2 + \|u(t,\cdot)\|_2^2. \quad (3.15)$$

Consider now the relation $z^\xi \leqq (z+1) \leqq (1+1/\eta)(z+\eta)$, which holds for all $z \geqq 0$, $\xi \in [0,1]$, $\eta > 0$, and take $z = \|u(t,\cdot)\|_2^{q_-}$, $\xi = 2/q_- < 1$, since $q_- > 2$ by (3.4), and $\eta = \mathscr{H}_0$, we obtain

$$\|u(t,\cdot)\|_2^2 \leqq (1 + 1/\mathscr{H}_0)(\|u(t,\cdot)\|_2^{q_-} + \mathscr{H}_0).$$

Of course the embedding $L^{q(\cdot)}(\Omega) \hookrightarrow L^2(\Omega)$ is continuous by (3.4), and so there exists a positive constant B, independent of u, such that $\|u(t,\cdot)\|_2 \leqq B\|u(t,\cdot)\|_{q(\cdot)}$. Combining the last two inequalities, we get

$$\|u(t,\cdot)\|_2^2 \leqq c_5\{\|u(t,\cdot)\|_{q(\cdot)}^{q_-} + \mathscr{H}(t)\}, \quad (3.16)$$

where $c_5 = (1 + 1/\mathscr{H}_0)\max\{1, B^{q_-}\} > 0$, being $\mathscr{H} \geqq \mathscr{H}_0$ in J by (3.8). Then, using (3.15) and (3.16) in the preceding estimate of $\mathscr{Z}'$, we find that

$$\begin{aligned}\mathscr{Z}' \geqq\ & k\left\{\lambda(1-r) - \ell^{-m'}\mathscr{H}_0^{r-\bar{r}}\right\}\mathscr{H}^{-r}\mathscr{H}' \\ & + \psi(\tilde{c}_3 - |\psi'/\psi|)\|u_t\|_2^2 + \psi\left\{\gamma p_+ - c_5|\psi'/\psi|\right\}\mathscr{H} \\ & + \psi\left\{\tilde{c}_2 - c_5|\psi'/\psi| - (c_4\ell)^m\mathscr{H}_0^{-\bar{r}}\right\}\|u(t,\cdot)\|_{q(\cdot)}^{q_-}.\end{aligned} \quad (3.17)$$

There is $T_1 \in J$ such that $2|\psi'/\psi| \leqq \min\{\tilde{c}_3, \gamma p_+/c_5, \tilde{c}_2/c_5\}$ for all $t \in J_1 = [T_1, \infty)$, since $\psi' = o(\psi)$ as $t \to \infty$. Then we take $\ell > 0$ so small that $4(c_4\ell)^m \leqq \tilde{c}_2\mathscr{H}_0^{\bar{r}}$ and $\lambda > 0$ so large that $\lambda \geqq \max\{\mathscr{H}_0^{r-\bar{r}}/\ell^{m'}(1-r),\ 1\}$ and $\mathscr{Z}(T_1) > 0$. In conclusion, we have shown that for almost all $t \in J_1$

$$\mathscr{Z}'(t) \geqq C\psi(t)\left\{\mathscr{H}(t) + \|u_t(t,\cdot)\|_2^2 + \|u(t,\cdot)\|_{q(\cdot)}^{q_-}\right\}, \quad (3.18)$$

where $2C = \min\{\tilde{c}_2/2, \tilde{c}_3, \gamma p_+\}$. Since $k(T_1), \mathscr{H}(T_1) > 0$, in particular $\mathscr{Z}(t) \geqq \mathscr{Z}(T_1) > 0$ for all $t \in J_1$.

On the other hand, from the definition of $\mathscr{Z}$, we obtain

$$\mathscr{Z}^\alpha \leqq \left(\lambda k \mathscr{H}^{1/\alpha} + \psi|\langle u_t, u\rangle|\right)^\alpha \leqq 2^{\alpha-1}\left\{(\lambda k)^\alpha \mathscr{H} + \psi^\alpha \|u_t\|_2^\alpha \|u\|_2^\alpha\right\}, \quad (3.19)$$

where $\alpha = 1/(1-r)$. Of course, $\alpha \in (1,2)$ by (3.13) and the choice of r. Put $\nu = 2/\alpha$, so that $\nu > 1$. Furthermore,

$$\frac{1}{\alpha\nu'} = \frac{\nu - 1}{\alpha\nu} = \frac{1}{\alpha} - \frac{1}{2} = \frac{1}{2} - r > \frac{1}{q_-}$$

by (3.13), and so $\alpha\nu' < q_-$. Thus, using the relation $z^\xi \leqq (z+1) \leqq (1+1/\eta)(\eta+z)$ once more, with $z = \|u(t,\cdot)\|_2^{q_-}$, $\xi = \alpha\nu'/q_- < 1$ and $\eta = \mathscr{H}_0$, it follows that

$$\|u(t,\cdot)\|_2^{\alpha\nu'} \leqq (1+1/\mathscr{H}_0)(\mathscr{H}_0 + \|u(t,\cdot)\|_2^{q_-}) \leqq c_5(\mathscr{H}(t) + \|u(t,\cdot)\|_{q(\cdot)}^{q_-}), \tag{3.20}$$

by (3.8), where c_5 is the same constant as in (3.16). Hence, from (3.19), Young's inequality and (3.20), for all $t \in J_1$

$$\begin{aligned}\mathscr{Z}(t)^\alpha &\leqq 2^{\alpha-1}[\max\{\lambda k(t), \psi(t)\}]^\alpha \left\{\mathscr{H}(t) + \|u_t(t,\cdot)\|_2^{\alpha\nu} + \|u(t,\cdot)\|_2^{\alpha\nu'}\right\} \\ &\leqq D[\max\{\lambda k(t), \psi(t)\}]^\alpha \left\{\mathscr{H}(t) + \|u_t(t,\cdot)\|_2^2 + \|u(t,\cdot)\|_{q(\cdot)}^{q_-}\right\},\end{aligned}$$

where $D = 2^{\alpha-1}(c_5+1)$. Combining this with (3.18) and $\lambda \geqq 1$, we obtain almost everywhere in J_1

$$\mathscr{Z}^{-\alpha}\mathscr{Z}' \geqq \frac{C}{D}\psi\,[\max\{\lambda k, \psi\}]^{-\alpha} \geqq c_6\psi\,[\max\{k, \psi\}]^{-\alpha},$$

where $c_6 = C/D\lambda^\alpha$. Finally, since $\alpha = 1+\theta$, being $r = \theta/(1+\theta)$, by $(3.6)_2$ we see that $\mathscr{Z}$ cannot be global. This completes the proof. □

Remark 3.2. In [17 and 23–25], assumptions $(\mathscr{F}_1)$–$(\mathscr{F}_3)$ and $(\mathcal{Q})$ are required in a stronger form and the main structure geometry there implies in particular

$$\inf_{t\in\mathbb{R}_0^+} \mathscr{A}u(t) \geqq w_1 \geqq 0, \quad Eu(0) < E_1 = \left(1 - \frac{\gamma p_+}{q_-}\right) w_1. \tag{3.21}$$

Observe that condition $(3.21)_1$ is always true and so $\mathscr{F}u(t) \geqq w_1 - Eu(0)$ as noted in Remark 3.1. Hence $w_2 > w_1 - E_1 = \gamma p_+ w_1/q_-$ when (3.21) holds. This yields $\tilde{E}_1 > E_1 \geqq 0$. Therefore (3.5) *is always weaker than* (3.21). Of course in (3.21) the case $w_1 > 0$ is more interesting, see [23], while the case $w_1 = 0$ was early treated in [17].

From the first part of Theorem 3.1 under $(\mathscr{F}_1)$–$(\mathscr{F}_2)$ it is evident that *if* $u \in K$ *is a solution of* (1.1) *in* $\mathbb{R}_0^+ \times \Omega$ *and* $w_2 = \inf_{t\in\mathbb{R}_0^+} \mathscr{F}u(t) \leqq 0$ *then* $Eu(0) \geqq \tilde{E}_1$. *Hence, being* $w_1 \leqq Eu(0) + w_2 \leqq Eu(0)$ *by* (3.1)*, also the case* (3.21) *can never occur, since* $E_1 < w_1$.

4. Applications of the main theorem

In this section we present useful consequences of Theorem 3.1 and a qualitative analysis, interesting in several applications. Suppose that M and f verify

$$M(\tau) = a + b\gamma\tau^{\gamma-1}, \quad a, b \geqq 0, \quad a+b > 0, \quad \gamma \begin{cases} > 1, & \text{if } b > 0, \\ = 1, & \text{if } b = 0, \end{cases} \tag{4.1}$$
$$f(t,x,u) = g(t,x)|u|^{\sigma(x)-2}u + c(x)|u|^{q(x)-2}u,$$

where $\sigma, q \in C_+(\overline{\Omega})$, $c \in L^\infty(\Omega)$ *is a non-negative function,* $g \in C(\mathbb{R}_0^+ \times \Omega)$ *is differentiable with respect to* t *and* $g_t \in C(\mathbb{R}_0^+ \times \Omega)$*; moreover*

$$\begin{aligned}
&\sigma_+ \leqq q_-, \ \max\{2, \gamma p_+\} < q_- \leqq q \leqq p^* \text{ in } \Omega, \quad \text{and} \quad c = \|c\|_\infty > 0;\\
&0 \leqq -g(t,x),\ g_t(t,x) \leqq h(x) \text{ in } \mathbb{R}_0^+ \times \Omega, \text{ for some } h \in L^1(\Omega),\\
&g(t,\cdot) \in L^{\wp(\cdot)}(\Omega) \text{ in } \mathbb{R}_0^+, \quad \text{where}\\
&\wp(x) = \begin{cases} q(x)/[q(x)-\sigma(x)], & \text{if } \sigma_+ < q_-,\\ \infty, & \text{if } \sigma_+ = q_-. \end{cases}
\end{aligned} \tag{4.2}$$

Lemma 4.1. *Assume that* M *and* f *verify* (4.1) *and* (4.2)*. Then* $(\mathscr{M})$*,* $(\mathscr{F}_1)$*,* $(\mathscr{F}_2)$ *and* $(\mathscr{F}_3)$*-*(i) *hold. Furthermore, if in addition*

$$\sigma_+ < q_- \quad \textit{and} \quad \overline{c} = \operatorname{ess\,inf}_{\overline{\Omega}} c(x) > 0, \tag{4.3}$$

then $(\mathscr{F}_3)$*-*(ii) *is verified, and in particular*

$$\langle f(t,\cdot,\phi(t,\cdot)), \phi(t,\cdot)\rangle \geqq q_- \mathscr{F}\phi(t) \quad \textit{for all } \phi \in K \quad \textit{and} \quad t \in \mathbb{R}_0^+. \tag{4.4}$$

Proof. Of course $(\mathscr{M})$ is satisfied. For any $\phi \in K$,

$$|(f(t,x,\phi(t,x)), \phi(t,x))| \leqq -g(t,x)|\phi(t,x)|^{\sigma(x)} + c\,|\phi(t,x)|^{q(x)},$$

so that by (4.2) we have $(f(t,x,\phi(t,x)), \phi(t,x)) \in L^1(\Omega)$ for all $t \in \mathbb{R}_0^+$ and $\langle f(t,\cdot,\phi(t,\cdot)), \phi(t,\cdot)\rangle \in L^1_{\text{loc}}(\mathbb{R}_0^+)$. Analogously, being

$$F(t,x,\phi(t,x)) = g(t,x)\frac{|\phi(t,x)|^{\sigma(x)}}{\sigma(x)} + c(x)\frac{|\phi(t,x)|^{q(x)}}{q(x)},$$

then also $F(t,x,\phi(t,x)) \in L^1(\Omega)$ for all $t \in \mathbb{R}_0^+$. Hence $(\mathscr{F}_1)$ holds. Furthermore, for any $\phi \in K$

$$\mathscr{F}\phi(t) = \mathscr{F}(t,\phi) = \int_\Omega \left\{ g(t,x)\frac{|\phi(t,x)|^{\sigma(x)}}{\sigma(x)} + c(x)\frac{|\phi(t,x)|^{q(x)}}{q(x)} \right\} \mathrm{d}x. \tag{4.5}$$

The same expression holds for $\mathscr{F}(t,w)$ when $(t,w) \in \mathbb{R}_0^+ \times W_0^{1,p(\cdot)}(\Omega)$. Thus, differentiation under the integral sign gives

$$\mathscr{F}_t(t,w) = \int_\Omega g_t(t,x)\,\frac{|w(x)|^{\sigma(x)}}{\sigma(x)}\,\mathrm{d}x.$$

Hence $\mathscr{F}_t \geqq 0$ in $\mathbb{R}_0^+ \times W_0^{1,p(\cdot)}(\Omega)$ by (4.2), and so $(\mathscr{F}_2)$ is fulfilled.

By (4.5) for all $\phi \in K$

$$\mathscr{F}\phi(t) \leqq \int_\Omega c(x)\frac{|\phi(t,x)|^{q(x)}}{q(x)}\,\mathrm{d}x \leqq \frac{c}{q_-}\,\rho_{q(\cdot)}(\phi(t,\cdot)), \tag{4.6}$$

being $g \leqq 0$. Hence, $(\mathscr{F}_3)$-(i) holds for all $\mathfrak{F} \geqq 0$, taking $c_1 = c/q_-$.

Assume now also (4.3) and put $\varepsilon_0 = q_- - \sigma_+ > 0$, so that for all $\varepsilon \in (0, \varepsilon_0)$ and $\phi \in K$, by (4.1)–(4.5), we have

$$\begin{aligned}&\langle f(t,\cdot,\phi(t,\cdot)),\phi(t,\cdot)\rangle - (q_- - \varepsilon)\mathscr{F}\phi(t)\\&\geqq \left(1 - \frac{q_- - \varepsilon}{\sigma_+}\right)\int_\Omega g(t,x)|\phi(t,x)|^{\sigma(x)}\,\mathrm{d}x + \frac{\varepsilon}{q_-}\int_\Omega c(x)|\phi(t,x)|^{q(x)}\,\mathrm{d}x\\&\geqq \frac{\bar{c}\varepsilon}{q_-}\,\rho_{q(\cdot)}(\phi(t,\cdot)).\end{aligned}$$

Thus, $(\mathscr{F}_3)$-(ii) is fulfilled, with $c_2 = \bar{c}\,\varepsilon/q_- > 0$ and $\bar{c}$ given in (4.3). Letting $\varepsilon \to 0$ in the above inequality, we get at once (4.4). □

In [23–25] assumptions (4.1)–(4.3) trivially hold, with $g \equiv 0$, $c \equiv 1$, $p \equiv 2$ and also q, γ constant with $q > 2\gamma$, and of course $\gamma = 1$ in [23,24].

Lemma 4.2. *Assume that the continuous damping function Q given in the Introduction verifies also the pointwise condition $(\mathcal{Q}_1)$ There exist constants $t_Q \geqq 0$, $m > 1$ and $\kappa \geqq 0$, with $m + \kappa < q_-$, and a non-negative function $d \in C(\mathbb{R}_0^+ \to L^{q_-/(q_- - \kappa - m)}(\Omega))$ such that*

$$|Q(t,x,u,v)| \leqq [d(t,x)|u|^\kappa]^{1/m}(Q(t,x,u,v),v)^{1/m'} \tag{4.7}$$

whenever $(t,x,u,v) \in [t_Q,\infty) \times \Omega \times \mathbb{R}^N \times \mathbb{R}^N$. Then $(\mathcal{Q})$ is satisfied along any solution u of the problem (1.1), with $T \geqq t_Q$ and $\delta(t) = \|d(t,\cdot)\|_{q_-/(q_- - \kappa - m)} \in C(\mathbb{R}_0^+)$, provided that $(\mathscr{F}_2)$ holds.

Proof. Clearly $\langle Q(t,\cdot,u(t,\cdot),u_t(t,\cdot)),u_t(t,\cdot)\rangle \geqq 0$ for each $t \geqq 0$ *along any solution $u \in K$ of the problem* (1.1), and so $\mathscr{D}u \geqq 0$, since also $\mathscr{F}_t u \geqq 0$ in $\mathbb{R}_0^+$ by $(\mathscr{F}_2)$, and $\mathscr{D}u$ is finite in $\mathbb{R}_0^+$ along u by (B)-(i).

By (4.7) and Hölder's inequality, for each $t \geqq t_Q$, *along any solution u* of (1.1),

$$\begin{aligned}&|Q(t,\cdot,u(t,\cdot),u_t(t,\cdot))\|_{q'_-}\\&\leqq \left(\int_\Omega \left\{d(t,x)|u(t,x)|^\kappa\right\}^{\frac{q_-}{q_- - m}}\,\mathrm{d}x\right)^{\frac{q_- - m}{mq_-}}\langle Q(t,\cdot,u,u_t),u_t(t,\cdot)\rangle^{1/m'},\end{aligned}$$

where $q'_- = (q_-)' = q_+/(q_+ - 1)$. On the other hand, applying once again Hölder's inequality, we find that

$$\int_\Omega \left\{d(t,x)|u(t,x)|^\kappa\right\}^{\frac{q_-}{q_- - m}}\,\mathrm{d}x \leqq \left(\int_\Omega d^{\frac{q_-}{q_- - m - \kappa}}\,\mathrm{d}x\right)^{\frac{q_- - m - \kappa}{q_- - m}}\left(\int_\Omega |u|^{q_-}\,\mathrm{d}x\right)^{\frac{\kappa}{q_- - m}}.$$

Hence, combining the last two inequalities, we get by $(\mathscr{F}_2)$

$$\begin{aligned}\|Q(t,\cdot,u(t,\cdot),u_t(t,\cdot))\|_{q'_-} &\leqq \|d(t,\cdot)\|^{\frac{1}{m}}_{\frac{q_-}{q_- - \kappa - m}}\|u(t,\cdot)\|^{\frac{\kappa}{m}}_{q_-}\langle Q(t,\cdot,u,u_t),u_t(t,\cdot)\rangle^{\frac{1}{m'}}\\&\leqq \delta(t)^{1/m}\|u(t,\cdot)\|^{\kappa/m}_{q_-}\mathscr{D}u(t)^{1/m'},\end{aligned}$$

where $\delta(t) = \|d(t,\cdot)\|_{q_-/(q_- -\kappa-m)} \in C(\mathbb{R}_0^+)$ by $(\mathcal{Q}_1)$, so that $\delta \in L^\infty_{\rm loc}(\mathbb{R}_0^+)$, while $q'_- = q_+/(q_+ - 1)$, as above. By the obvious continuity of the embedding $L^{q(\cdot)}(\Omega) \hookrightarrow L^{q_-}(\Omega)$ there is a constant $B_1 > 0$ independent of u such that $\|u(t,\cdot)\|_{q_-} \leqq B_1\|u(t,\cdot)\|_{q(\cdot)}$ for all $t \in \mathbb{R}_0^+$. Hence

$$\begin{aligned}|\langle Q(t,\cdot,u,u_t),u(t,\cdot)\rangle| &\leqq \|Q(t,\cdot,u,u_t)\|_{q'_-}\|u(t,\cdot)\|_{q_-}\\ &\leqq q_1\delta(t)^{1/m}\|u(t,\cdot)\|_{q(\cdot)}^{1+\kappa/m}\mathscr{D}u(t)^{1/m'},\end{aligned} \tag{4.8}$$

where $q_1 = B_1^{1+\kappa/m}$. □

Of course in [23–25] assumption $(\mathcal{Q}_1)$ is automatic, with $\kappa \equiv 0$ and $1 < m < q$, being there $q_- = q_+ = q$.

Let us now distinguish two cases, depending on whether b is zero or not. Of course, when $b > 0$ and $a = 0$ we are in the so called *degenerate case*, which is in our context more interesting. Recall that we have assumed $\gamma > 1$ when $b > 0$, while $\gamma = 1$ if $b = 0$, and put $s = b$ if $b > 0$, while $s = a$ if $b = 0$.

Lemma 4.3. *Assume* (4.1) *and* (4.2). *If $u \in K$ is a solution of* (1.1) *in $\mathbb{R}_0^+ \times \Omega$, then for all $t \in \mathbb{R}_0^+$*

$$Eu(t) \geqq \frac{s}{(\Lambda p_+)^\gamma}\min\left\{\upsilon(t)^{p_-},\upsilon(t)^{p_+}\right\}^\gamma - \frac{c}{q_-}\max\{\upsilon(t)^{q_-},\upsilon(t)^{q_+}\},$$

where $\upsilon(t) = \|u(t,\cdot)\|_{q(\cdot)}$,

$$\Lambda = \max\left\{\lambda_{q(\cdot)}^{p_+},\lambda_{q(\cdot)}^{p_-},(s\gamma/cp_+^{\gamma-1})^{1/\gamma}\right\} \tag{4.9}$$

and $\lambda_{q(\cdot)}$ is the constant introduced in (2.3).

Proof. Let $u \in K$ be a solution of (1.1) in $\mathbb{R}_0^+ \times \Omega$. By (2.1) and (2.3) we have

$$\begin{aligned}\mathscr{A}u(t) &\geqq \mathscr{M}(\mathscr{I}u(t)) \geqq \frac{a}{p_+}\rho_{p(\cdot)}(Du(t,\cdot)) + \frac{b}{p_+^\gamma}\left[\rho_{p(\cdot)}(Du(t,\cdot))\right]^\gamma\\ &\geqq \frac{a}{p_+}\min\left\{\|Du(t,\cdot)\|_{p(\cdot)}^{p_-},\|Du(t,\cdot)\|_{p(\cdot)}^{p_+}\right\}\\ &\quad+\frac{b}{p_+^\gamma}\min\left\{\|Du(t,\cdot)\|_{p(\cdot)}^{\gamma p_-},\|Du(t,\cdot)\|_{p(\cdot)}^{\gamma p_+}\right\}\\ &\geqq \frac{a}{\Lambda p_+}\min\left\{\|u(t,\cdot)\|_{q(\cdot)}^{p_-},\|u(t,\cdot)\|_{q(\cdot)}^{p_+}\right\}\\ &\quad+\frac{b}{(\Lambda p_+)^\gamma}\min\left\{\|u(t,\cdot)\|_{q(\cdot)}^{\gamma p_-},\|u(t,\cdot)\|_{q(\cdot)}^{\gamma p_+}\right\}\\ &\geqq \frac{s}{(\Lambda p_+)^\gamma}\min\left\{\upsilon(t)^{p_-},\upsilon(t)^{p_+}\right\}^\gamma.\end{aligned} \tag{4.10}$$

Therefore, since $Eu(t) \geqq \mathscr{A}u(t) - \mathscr{F}u(t)$ for each $t \in \mathbb{R}_0^+$ by (3.1), the assertion follows at once by (4.10) and (4.6). □

From Lemma 4.3 we obtain

$$Eu(t) \geqq \varphi(\upsilon(t)) \quad \text{for all} \quad t \in \mathbb{R}_0^+, \tag{4.11}$$

where $\varphi : \mathbb{R}_0^+ \to \mathbb{R}$ is defined by $\varphi(\upsilon) = \varphi_1(\upsilon)$ if $\upsilon \in [0, 1]$, while $\varphi(\upsilon) = \varphi_2(\upsilon)$ if $\upsilon \geqq 1$, with

$$\varphi_1(\upsilon) = \frac{s}{(\Lambda p_+)^\gamma} \upsilon^{\gamma p_+} - \frac{c}{q_-} \upsilon^{q_-}, \quad \varphi_2(\upsilon) = \frac{s}{(\Lambda p_+)^\gamma} \upsilon^{\gamma p_-} - \frac{c}{q_-} \upsilon^{q_+}.$$

It is easy to see that φ attains its maximum at

$$\upsilon_1 = a_1^{1/(q_- - \gamma p_+)}, \quad \text{where} \quad a_1 = \frac{s\gamma p_+}{c(\Lambda p_+)^\gamma}. \tag{4.12}$$

The choice of Λ in (4.9) guarantees that $\upsilon_1 \in (0, 1]$. Clearly φ_2 takes its maximum at $\upsilon_2 = a_2^{1/(q_+ - \gamma p_-)}$, where $a_2 = p_- q_- a_1 / p_+ q_+ \leqq a_1 \leqq 1$. Hence φ is strictly decreasing for $\upsilon \geqq \upsilon_1$, with $\varphi(\upsilon) \to -\infty$ as $\upsilon \to \infty$. Finally,

$$\varphi(\upsilon_1) = \left(1 - \frac{\gamma p_+}{q_-}\right) w_1 = E_1 > 0, \quad \text{where} \quad w_1 = \frac{s\upsilon_1^{\gamma p_+}}{(\Lambda p_+)^\gamma} > 0. \tag{4.13}$$

Put

$$\Sigma = \{(\upsilon, E) \in \mathbb{R}^2 : \upsilon > \upsilon_1, \ E < E_1\}.$$

Theorem 4.1. *Assume* (4.1), (4.2) *and* ($\mathcal{Q}_1$). *If* $u \in K$ *is a solution of* (1.1) *in* $\mathbb{R}_0^+ \times \Omega$, *then* $w_2 = \inf_{t \in \mathbb{R}_0^+} \mathscr{F}u(t) > -\infty$. *If, furthermore,* $Eu(0) < \tilde{E}_1$, *with* $\tilde{E}_1$ *given in* (3.5), *then* $w_2 > 0$ *and* $(\upsilon(t), Eu(t)) \in \tilde{\Sigma}$ *for all* $t \in \mathbb{R}_0^+$, *where*

$$\tilde{\Sigma} = \{(\upsilon, E) \in \mathbb{R}^2 : \upsilon > \upsilon_1, \ E < \tilde{E}_1\}, \tag{4.14}$$

and υ_1 *is defined in* (4.12). *Consequently, if in addition* (4.3) *holds, then there are no solutions* $u \in K$ *of the problem* (1.1) *in* $\mathbb{R}_0^+ \times \Omega$, *with* $Eu(0) < \tilde{E}_1$, *for which there exist positive functions* ψ, k *verifying* (3.6)–(3.7) *as in Theorem* 3.1.

Proof. Clearly Lemmas 4.1 and 4.2 are available, so that assumptions ($\mathscr{F}_1$), ($\mathscr{F}_2$), ($\mathscr{F}_3$)-(i) and ($\mathcal{Q}$) of Theorem 3.1 are satisfied along any solution u of (1.1). The fact that w_2 is finite and positive are an immediate consequence of Theorem 3.1. By ($\mathscr{F}_2$), ($\mathcal{Q}_1$) and (B)-(ii) clearly $Eu(t) \leqq Eu(0) < \tilde{E}_1$ for all $t \in \mathbb{R}_0^+$.

Suppose now that there exists $t_1 \in \mathbb{R}_0^+$ such that $\upsilon(t_1) \leqq \upsilon_1$. Then, by (4.6) and $(2.1)_2$ we have $w_2 \leqq \mathscr{F}u(t_1) \leqq c\upsilon(t_1)^{q_-}/q_-$. On the other hand, $\mathscr{A}u(t_1) \geqq s\upsilon(t_1)^{\gamma p_+}/(\Lambda p_+)^\gamma$ by (4.10). Now, by (3.1), ($\mathscr{F}_2$), ($\mathcal{Q}_1$) and (B)-(ii), it follows that

$$\begin{aligned}\left(\frac{q_-}{\gamma p_+} - 1\right) \frac{c}{q_-} \upsilon(t_1)^{q_-} &\geqq \tilde{E}_1 > Eu(0) \geqq \mathscr{A}u(t_1) - \mathscr{F}u(t_1) \\ &\geqq \frac{s}{(\Lambda p_+)^\gamma} \upsilon(t_1)^{\gamma p_+} - \frac{c}{q_-} \upsilon(t_1)^{q_-}.\end{aligned}$$

That is $\upsilon(t_1) > \left[s\gamma p_+ / c(\Lambda p_+)^\gamma\right]^{1/(q_- - \gamma p_+)} = \upsilon_1$ by (4.12). This is an obvious contradiction. Therefore $\upsilon(t) > \upsilon_1$ for all $t \in \mathbb{R}_0^+$, and $(\upsilon(t), Eu(t)) \in \tilde{\Sigma}$ for all $t \in \mathbb{R}_0^+$, as required.

The last part of the theorem is again a direct consequence of Theorem 3.1. □

If $\mu = 0$, f is as in (4.1)–(4.3), with $g(t,x) = g(x)$, and $Q(t,x,u,0) = 0$ in $(\mathcal{Q}_1)$, then *any stationary solution* $u = u(x)$ *of* (1.1)*, with* $w_2 = \mathscr{F}u > 0$*, has the property that* $Eu \geqq \tilde{E}_1$*, that is Theorem* 4.1 *can never be applied*. Indeed, in this case Eu, $\mathscr{A}u$ and $\mathscr{F}u > 0$ are constant in t. Hence by (3.1), (3.3) and (4.4), since $\langle Au, u\rangle = \langle f(\cdot,u), u\rangle$ being u a stationary solution of (1.1),

$$Eu = \left(\frac{\mathscr{A}u}{\mathscr{F}u} - 1\right)\mathscr{F}u \geqq \left(\frac{q_-}{\gamma p_+} - 1\right)\mathscr{F}u = \left(\frac{q_-}{\gamma p_+} - 1\right) w_2 = \tilde{E}_1,$$

as claimed.

In special reverse situations in which the external force f is of restoring type, it is possible to show that if $\upsilon(0) < \upsilon_1$ and $Eu(0) < E_1$, then $\upsilon(t) < \upsilon_1$ and $Eu(t) < E_1$ for all $t \in \mathbb{R}_0^+$, that is any point $(\upsilon(t), Eu(t))$ on the trajectory of a global strong solution $u \in K$ must remain in the potential well, see, for example, [3, Lemma 4.3] and for dissipative wave systems [21, Remark on page 45], as well as the references therein.

The global nonexistence results, given in [17] and [23–25] *in special subcases of this paper, concern only the region* Σ *which is smaller than* $\tilde{\Sigma}$. Indeed, if $Eu(0) < \tilde{E}_1$, under the assumptions of Theorem 4.1, then for all $t \geqq 0$

$$\upsilon(t) > \upsilon_1, \quad \mathscr{A}u(t) > \frac{s}{(\Lambda p_+)^\gamma}\min\left\{\upsilon_1^{p_-}, \upsilon_1^{p_+}\right\}^\gamma = w_1 > 0, \tag{4.15}$$

by (4.10) and (4.13). Hence, $E_1 < \tilde{E}_1$ as already proved in Remark 3.2. In any case we present also new results under the assumption $Eu(0) \leqq E_1$, the first of which being the following

Theorem 4.2. *Assume* (4.1)*,* (4.2) *and* $(\mathcal{Q}_1)$*. Let* $u \in K$ *be a solution of* (1.1) *in* $\mathbb{R}_0^+ \times \Omega$*, such that* $Eu(0) < E_1$*, with* E_1 *given in* (4.13)*. Then* $\upsilon_1 \notin \overline{\upsilon(\mathbb{R}_0^+)}$ *and* $w_2 = \inf_{t\in\mathbb{R}_0^+}\mathscr{F}u(t) \neq \gamma p_+ w_1/q_-$*, where* υ_1 *and* w_1 *are defined in* (4.12) *and* (4.13)*, respectively. Moreover,* $w_2 > \gamma p_+ w_1/q_-$ *if and only if* $\overline{\upsilon(\mathbb{R}_0^+)} \subset (\upsilon_1, \infty)$.

Proof. Let $u \in K$ be a solution of (1.1) in $\mathbb{R}_0^+ \times \Omega$. Then $w_2 > -\infty$, as shown in the first part of Theorem 3.1. Assume also that $Eu(0) < E_1$. We first claim that $\upsilon_1 \notin \overline{\upsilon(\mathbb{R}_0^+)}$. Proceed by contradiction and suppose that $\upsilon_1 \in \overline{\upsilon(\mathbb{R}_0^+)}$. It follows that there exists a sequence $(t_j)_j$ in $\mathbb{R}_0^+$ such that $\upsilon(t_j) \to \upsilon_1$ as $j \to \infty$. By (4.11) we have $E_1 > Eu(0) \geqq Eu(t_j) \geqq \varphi(\upsilon(t_j))$, which provides $E_1 > E_1$ by the continuity of $\varphi \circ \upsilon$, and the claim is proved.

We show that $w_2 \neq \gamma p_+ w_1/q_-$. Otherwise, $\mathscr{F}u(t) \geqq \gamma p_+ w_1/q_-$ for all $t \in \mathbb{R}_0^+$. Therefore, by (3.1) and (4.13), we have

$$\left(\frac{q_-}{\gamma p_+} - 1\right)\mathscr{F}u(t) \geqq E_1 > Eu(0) \geqq \mathscr{A}u(t) - \mathscr{F}u(t),$$

so that, using (4.6) and (4.10), for each $t \in \mathbb{R}_0^+$ we get

$$\frac{c}{\gamma p_+}\rho_{q(\cdot)}(u(t,\cdot)) \geqq \frac{q_-}{\gamma p_+}\mathscr{F}u(t) > \mathscr{A}u(t) \geqq \frac{s}{(\Lambda p_+)^\gamma}\min\left\{\upsilon(t)^{p_-}, \upsilon(t)^{p_+}\right\}^\gamma.$$

Hence, if $t \in \mathbb{R}_0^+$ and $\upsilon(t) \leqq 1$, then $c\upsilon(t)^{q_-}/\gamma p_+ > s\upsilon(t)^{\gamma p_+}/(\Lambda p_+)^\gamma$ using also (2.1), that is $\upsilon(t) > \upsilon_1$. On the other hand, if $\upsilon(t) > 1$, then automatically $\upsilon(t) > \upsilon_1$, being $\upsilon_1 \leqq 1$. Hence, $\upsilon(t) > \upsilon_1$ for each $t \in \mathbb{R}_0^+$, so that by (4.10), we immediately obtain $\mathscr{A}u(t) > w_1$ for all $t \in \mathbb{R}_0^+$, where w_1 is defined in (4.13). Consequently, $\mathscr{F}u(t) \geqq w_1 - Eu(0) > w_1 - E_1 = \gamma p_+ w_1/q_-$ for all $t \in \mathbb{R}_0^+$ and in turn $w_2 > \gamma p_+ w_1/q_-$. This gives an obvious contradiction.

Suppose that $w_2 > \gamma p_+ w_1/q_-$. We prove that $\upsilon(\mathbb{R}_0^+) \subset (\upsilon_1, \infty)$, which immediately gives $\overline{\upsilon(\mathbb{R}_0^+)} \subset (\upsilon_1, \infty)$, since $\upsilon_1 \notin \overline{\upsilon(\mathbb{R}_0^+)}$. Assume by contradiction that there exists $t_1 \in \mathbb{R}_0^+$ for which $\upsilon(t_1) \leqq \upsilon_1$. It follows

$$\mathscr{F}u(t_1) \leqq \frac{c}{q_-}\upsilon(t_1)^{q_-} \leqq \frac{c}{q_-}\upsilon_1^{q_-} = \frac{\gamma p_+}{q_-}w_1,$$

so that $w_2 \leqq \gamma p_+ w_1/q_-$, which is impossible.

On the other hand, if $\overline{\upsilon(\mathbb{R}_0^+)} \subset (\upsilon_1, \infty)$, then $\upsilon(t) > \upsilon_1$ for all $t \in \mathbb{R}_0^+$. Hence $\mathscr{F}u(t) \geqq w_1 - Eu(0) > w_1 - E_1 = \gamma p_+ w_1/q_-$ for all $t \in \mathbb{R}_0^+$ by (3.1) and in turn $w_2 > \gamma p_+ w_1/q_-$, as required. □

In the next corollary we present an application of both Theorems 3.1 and 4.1. In particular, we provide sufficient conditions under which assumptions (3.6)–(3.7) of Theorem 3.1 are satisfied. *Let $Q = Q(t,x,u,v)$ be a continuous damping function* as in the Section 1 and assume also that *there exists $t^* \gg 1$ such that for all $(t,x,u,v) \in [t^*, \infty) \times \Omega \times \mathbb{R}^N \times \mathbb{R}^N$*

$$Q(t,x,u,v) = d(t,x)|u|^\kappa |v|^{m-2}v, \tag{4.16}$$

where $\kappa \geqq 0$, $m > 1$, $m + \kappa < q_-$, $d \in C(\mathbb{R}_0^+ \to L^{q_-/(q_- - \kappa - m)}(\Omega))$, with $d(t,x) \geqq 0$ in $\mathbb{R}_0^+ \times \Omega$. Put $\delta(t) = \|d(t,\cdot)\|_{q_-/(q_- - \kappa - m)}$. Hence,

$$|Q(t,x,u,v)| = [d(t,x)|u|^\kappa]^{1/m}[(Q(t,x,u,v),v)]^{1/m'}$$

for all $(t,x,u,v) \in [t^*, \infty) \times \Omega \times \mathbb{R}^N \times \mathbb{R}^N$, so that $(\mathcal{Q}_1)$ holds with $t_Q = t^*$. Now put $J = [t^*, \infty)$.

Corollary 4.1. *Assume* (4.1)–(4.3), (4.16) *and that $\delta(t) \leqq \delta_1(1+t)^\ell$ for each $t \in J$, for some appropriate numbers $\delta_1 \geqq 1$ and $\ell \leqq m - 1$. Then there are no solutions $u \in K$ of* (1.1) *in $\mathbb{R}_0^+ \times \Omega$, with $Eu(0) < \tilde{E}_1$.*

Proof. Let $u \in K$ be a solution of (1.1) in $\mathbb{R}_0^+ \times \Omega$, with $Eu(0) < \tilde{E}_1$. All the structural assumptions of Theorem 4.1 are available, and it remains to provide the auxiliary functions k and ψ verifying (3.6), with θ_0 as in (3.7) to reach the desired contradiction.

Take $k(t) \equiv \delta_1^{m'}$ and $\psi(t) = \delta_1(1+t)^{-\ell/(m-1)}$ for each $t \in J$, so that $(3.6)_1$ is verified in J. If $\ell \geqq 0$, then $k(t) \geqq \psi(t)$ for each $t \in J$, being $\delta_1 \geqq 1$. Otherwise, if $\ell < 0$, take $S_0 \geqq \max\{t^*, \delta_1^{1/|\ell|} - 1\}$ so that for all $t \geqq S_0$

$$\psi(t) \geqq \psi(S_0) = \delta_1(1+S_0)^{|\ell|/(m-1)} \geqq \delta_1^{m'} = k(t).$$

Hence, for each $t \geqq S_0$ we have

$$\psi(t)[\max\{k(t), \psi(t)\}]^{-(1+\theta)} = \begin{cases} \delta_1^{1-m'(1+\theta)}(1+t)^{-\ell/(m-1)}, & \text{if } \ell \geqq 0, \\ \delta_1^{-\theta}(1+t)^{\ell\theta/(m-1)}, & \text{if } \ell < 0. \end{cases}$$

Clearly $(3.6)_2$ is verified for all $\theta \in (0, \theta_0)$, with θ_0 as in (3.7), whenever $\ell \geqq 0$ since $\ell \leqq m-1$. While, if $\ell < 0$, then we choose $\theta \in (0, \theta_0)$, so small that $\theta \leqq (m-1)/|\ell|$, that is so small that $(3.6)_2$ holds. □

Observe that even in the case $\ell < 0$ the function δ in Corollary 4.1 does not need to be non-increasing in $[S_0, \infty)$, as the function $\delta(t) = \delta_1|\sin t|(1+t)^\ell$ shows. As a matter of fact, δ obviously verifies the relation $\delta(t) \leqq \delta_1(1+t)^\ell$, but it approaches zero as $t \to \infty$ oscillating.

From now on in the section *we assume* for simplicity *the structure assumptions* (4.1)–(4.3) *and* (4.16), *with* $\delta(t) \leqq \delta_1(1+t)^\ell$ *for each* $t \in J$ *and some* $\delta_1 \geqq 1$, *with* $\ell \leqq m-1$.

The following corollary extends and generalizes Theorems 5 and 6 of [17], the first part of Theorem 4 of [23] and Theorem 3.1 of [24], and Theorems 4.3 and 4.5 of [25].

Corollary 4.2. *Problem* (1.1) *does not possess solutions* $u \in K$ *in* $\mathbb{R}_0^+ \times \Omega$, *with*

$$\|u(0,\cdot)\|_{q(\cdot)} > \upsilon_1, \quad Eu(0) < E_1, \tag{4.17}$$

where E_1 *is defined in* (4.13).

Proof. Assume by contradiction that $u \in K$ is a solution of (1.1) in $\mathbb{R}_0^+ \times \Omega$, verifying (4.17). By Theorem 4.2 then $w_2 > \gamma p_+ w_1/q_-$. Hence $Eu(0) < E_1 < \tilde{E}_1$ and the contradiction follows at once by an application of Corollary 4.1. □

Proposition 4.1. *If* $u \in K$ *is a solution of* (1.1) *in* $\mathbb{R}_0^+ \times \Omega$, *with* $Eu(0) \leqq E_1$, *where* E_1 *is defined in* (4.13), *then*

$$w_2 \leqq \frac{\gamma p_+}{q_-} w_1. \tag{4.18}$$

Proof. Otherwise $w_2 > \gamma p_+ w_1/q_-$, so that $Eu(0) < \tilde{E}_1$, and u could not be global by Corollary 4.1. □

In the sequel of the section we assume also

($\mathscr{D}$) *There exists* $t_* > 0$ *such that either*

(i) $g_t(t,x) \geqq g_0(t) > 0$ *for each* $(t,x) \in [0, t_*) \times \Omega$, *or*

(ii) $\phi \in K$ *and* $\langle Q(t,\cdot,\phi,\phi_t), \phi_t\rangle = 0$ *in* $[0, t_*]$ *implies either* $\phi(t,\cdot) \equiv 0$ *or* $\phi_t(t,\cdot) \equiv 0$ *for all* $t \in [0, t_*]$,

which allows us to extend and generalizes the second part of Theorems 4 of [23] and 3.1 of [24].

Theorem 4.3. *Problem* (1.1) *does not possess solutions* $u \in K$ *in* $\mathbb{R}_0^+ \times \Omega$, *with*

$$\|u(0,\cdot)\|_{q(\cdot)} > \upsilon_1, \qquad Eu(0) = E_1. \tag{4.19}$$

Proof. Assume by contradiction that $u \in K$ is a global solution of (1.1) in $\mathbb{R}_0^+ \times \Omega$, verifying (4.19). By Proposition 4.1 we have $w_2 \leqq \gamma p_+ w_1/q_-$. We first claim that $w_2 < \gamma p_+ w_1/q_-$ cannot occur. Otherwise there exists t_0 such that $\mathscr{F}u(t_0) < \gamma p_+ w_1/q_-$, so that by (B)-(ii) and (3.1)

$$w_1 - \mathscr{F}u(t_0) > E_1 = Eu(0) \geqq Eu(t_0) \geqq \mathscr{A}u(t_0) - \mathscr{F}u(t_0),$$

and by (4.10), (4.12) and (4.13) it is not hard to see that $\upsilon(t_0) < \upsilon_1$. Hence $t_0 > 0$ by (4.19) and by the continuity of υ there exists $s \in (0, t_0)$ such that $\upsilon(s) = \upsilon_1$. The above argument and (4.6) show that $E_1 = Eu(0) \geqq Eu(s) \geqq w_1 - c\upsilon_1^{q_-}/q_- = E_1$. In other words, $Eu(s) = E_1$ and $\int_0^s \mathscr{D}u(\tau)\,\mathrm{d}\tau = 0$ by (B)–(ii). Consequently $\mathscr{D}u \equiv 0$ in $[0, s]$ and so, by $(\mathscr{F}_2)$ and (4.16), we obtain $\langle Q(t, \cdot, u(t, \cdot), u_t(t, \cdot)), u_t(t, \cdot)\rangle = 0$ and $\mathscr{F}_t u(t) = 0$ for all $t \in [0, s]$.

Now, if $(\mathscr{D})$-(i) holds, then

$$0 = \mathscr{F}_t u(t) = \int_\Omega g_t(t, x)\frac{|u(t, x)|^{\sigma(x)}}{\sigma(x)}\,\mathrm{d}x \geqq \frac{g_0(t)}{\sigma_+}\rho_{\sigma(\cdot)}(u(t, \cdot)) \geqq 0$$

for each $t \in [0, s_0]$, where $s_0 = \min\{t_*, s\}$. Therefore $\rho_{\sigma(\cdot)}(u(t, \cdot)) \equiv 0$ and in turn $u \equiv 0$ in $[0, s_0] \times \Omega$, by (2.1). But this occurrence is impossible, since $\|u(0, \cdot)\|_{q(\cdot)} = \upsilon(0) > \upsilon_1 > 0$ by $(4.19)_1$, so that we reach a contradiction.

While, if $(\mathscr{D})$-(ii) holds, since $\langle Q(t, \cdot, u(t, \cdot), u_t(t, \cdot)), u_t(t, \cdot)\rangle = 0$ for all $t \in [0, s_0]$, we get that either $u(t, \cdot) = 0$ or $u_t(t, \cdot) = 0$ for all $t \in [0, s_0]$, where as above $s_0 = \min\{t_*, s\}$. Again, as already shown, the first case cannot occur since $\upsilon(0) > \upsilon_1$. In the latter, u is clearly constant with respect to t in $[0, s_0]$, and so $u(t, x) = u(0, x)$ for each $t \in [0, s_0]$. Taking $\phi(t, x) = u(0, x)$ in the *Distribution Identity* (A), then for each $t \in [0, s_0]$ we have $t\langle Au(0, \cdot), u(0, \cdot)\rangle = \int_0^t \langle f(\tau, \cdot, u(0, \cdot)), u(0, \cdot)\rangle\,\mathrm{d}\tau$, since $\langle Q(t, \cdot, u(0, \cdot), 0), u(0, \cdot)\rangle = 0$ by (4.8), being $\mathscr{D}u = 0$ in $[0, s_0]$. Therefore $\langle Au(0, \cdot), u(0, \cdot)\rangle = \langle f(t, \cdot, u(0, \cdot)), u(0, \cdot)\rangle$ for each $t \in [0, s_0]$, and so $\langle A(u(0, \cdot)), u(0, \cdot)\rangle = \langle f(0, \cdot, u(0, \cdot)), u(0, \cdot)\rangle$. Now $\gamma p_+ \mathscr{A}u(0) \geqq q_- \mathscr{F}u(0)$ by (3.3) and $(\mathscr{F}_3)$. On the other hand, $E_1 = Eu(0) = \mathscr{A}u(0) - \mathscr{F}u(0)$ by (3.1), since $u_t(0, \cdot) = 0$. By (4.15) and (4.13) we have $\mathscr{A}u(0) > w_1 > 0$, and so

$$E_1 \geqq \left(1 - \frac{\gamma p_+}{q_-}\right)\mathscr{A}u(0) > \left(1 - \frac{\gamma p_+}{q_-}\right)w_1 = E_1$$

by (4.13). This contradiction shows the claim.

Hence $w_2 = \gamma p_+ w_1/q_-$. In particular $\mathscr{F}u(t) \geqq \gamma p_+ w_1/q_-$ for all $t \in \mathbb{R}_0^+$ and we assert that equality cannot occur at a finite time. Indeed, if there is s such that $\mathscr{F}u(s) = \gamma p_+ w_1/q_-$, then $\upsilon(s) \geqq \upsilon_1$ by (4.6) and (2.1). But $\upsilon(s) > \upsilon_1$ would imply $Eu(0) > E_1$, contradicting (4.19). Hence $\mathscr{F}u(s) = \gamma p_+ w_1/q_-$, $\upsilon(s) = \upsilon_1$ and so $Eu(s) = E_1$. From now on we can repeat the argument above in correspondence at such s and assumption $(\mathscr{D})$ will produce the required contradiction again.

Therefore it remains to consider the case $w_2 = \gamma p_+ w_1/q_-$, $\mathscr{F}u(t) > w_2$ and $\upsilon(t) > \upsilon_1$ for all $t \in \mathbb{R}_0^+$. A continuity argument shows at once that

$$\liminf_{t\to\infty} \mathscr{F}u(t) = w_2.$$

On the other hand by (3.1) and (B)-(ii) we have $w_1 - \mathscr{F}u(t) < Eu(t) \leqq E_1$, so that $\limsup_{t\to\infty} Eu(t) = E_1$. Hence $\int_0^\infty \mathscr{D}u(\tau)\,d\tau = 0$ by monotonicity. In particular $\mathscr{D}u \equiv 0$ in $\mathbb{R}_0^+$, which is again impossible by ($\mathscr{D}$) using the argument already produced. This completes the proof. □

Remark 4.1. The possibility to cover the case (4.19) was first discovered by VITILLARO [23], but only for *strong* solutions. Indeed, to cover (4.19) in [23,24] the energy conservation law was considered with the equality sign, as in $(B)_s$-(ii), even if not explicitly stated, as for example, in the proof of case (a) of Theorem 3 of [23]. Furthermore, in [23,24] essentially the autonomous case was treated. It is interesting to note that Theorems 3 and 4 of [23] covers only the case when $Q > 0$ near 0, while in Theorem 5.4 we are able to consider even the situation in which $Q \equiv 0$, provided that ($\mathscr{D}$)-(i) is valid. The proof of Theorem 4.3 differs from that of Theorems 3 and 4 of [23], being based on Proposition 4.1 and on Theorem 3.1.

Appendix: The linear dissipation case

In this section we provide a non-continuation result for (1.1) when ($\mathscr{F}_1$) and ($\mathscr{F}_2$) hold as in Section 3, while ($\mathscr{F}_3$) is replaced by a weaker condition provided that the damping term is linear, that is $Q(t,x,u,v) = Q(t)v$. More precisely, (1.1) reduces simply to

$$\begin{cases} u_{tt} - M\left(\mathscr{I}u(t)\right)\Delta_{p(x)}u + \mu|u|^{p(x)-2}u + Q(t)u_t = f(t,x,u), \\ u(t,x) = 0 \quad \text{on } \mathbb{R}_0^+ \times \partial\Omega, \end{cases} \tag{5.1}$$

where as before $u = (u_1, \ldots, u_N) = u(t,x)$ is the vectorial displacement, $N \geqq 1$, the domain Ω is bounded in $\mathbb{R}^n$ and $\mu \geqq 0$. In place of ($\mathcal{Q}$) *we assume throughout the section the stronger condition*

(Q) $$Q \in C^1(\mathbb{R}_0^+), \quad \textit{with} \quad Q, -Q' \geqq 0,$$

which indeed implies ($\mathcal{Q}_1$), see the next Remark 5.1.

The function space K is given in Section 2, while $E\phi$, $\mathscr{A}\phi$ and $A\phi$, with $\phi \in K$, are the same functions introduced in (3.1) and (3.2). As in Section 3 we define a *solution of* (5.1) as a function $u \in K$ satisfying the two conditions:

(A) *Distribution Identity*

$$\langle u_t, \phi\rangle\,]_0^t = \int_0^t \{\, \langle u_t, \phi_t\rangle - M(\mathscr{I}u(\tau))\langle |Du|^{p(\cdot)-2}Du, D\phi\rangle \\ -\mu\langle |u|^{p(\cdot)-2}u, \phi\rangle - \langle Q(\tau)u_t - f, \phi\rangle\}\,d\tau$$

for all $t \in \mathbb{R}_0^+$ and $\phi \in K$;

(B) *Energy Conservation*

$$Eu(t) \leqq Eu(0) - \int_0^t \left\{ Q(\tau)\|u_t(\tau,\cdot)\|_2^2 + \mathscr{F}_t u(\tau) \right\} d\tau$$

for all $t \in \mathbb{R}_0^+$.

By $(\mathscr{F}_1)$, $(\mathscr{F}_2)$ and (Q) properties (A) and (B) are meaningful. Instead of $(\mathscr{F}_3)$ here we assume the weaker assumption: *there exists* $q \in C_+(\overline{\Omega})$*, satisfying* (3.4)*, such that*

$(\mathscr{F}_3)'$ $\quad \langle f(t,\cdot,\phi(t,\cdot)), \phi(t,\cdot)\rangle \geqq q_- \mathscr{F}\phi(t)$ *for* $(t,\phi) \in \mathbb{R}_0^+ \times K$.

See (4.4).

Theorem 5.1. *Suppose that* $(\mathscr{M})$, (Q), $(\mathscr{F}_1)$ *and* $(\mathscr{F}_2)$ *hold true. If* $u \in K$ *is a solution of* (5.1) *in* $\mathbb{R}_0^+ \times \Omega$*, then* $w_2 = \inf_{t\in\mathbb{R}_0^+} \mathscr{F}u(t) > -\infty$.

Finally, assume that $2 < \gamma p_+$ *and also that* $(\mathscr{F}_3)'$ *is satisfied. Then there are no solutions* $u \in K$ *of* (5.1) *in* $\mathbb{R}_0^+ \times \Omega$*, with* $Eu(0) < \tilde{E}_1$*, where* $\tilde{E}_1$ *is defined in* (3.5).

Proof. The first part of the result is just a direct consequence of Theorem 3.1, since clearly here $(Q(t)v, v) = Q(t)|v|^2 \geqq 0$ for all $(t,v) \in \mathbb{R}_0^+ \times \mathbb{R}^N$.

Assume now that $2 < \gamma p_+$ and also $(\mathscr{F}_3)'$ holds, and by contradiction that there exists a solution $u \in K$ of (5.1) in $\mathbb{R}_0^+ \times \Omega$, with $Eu(0) < \tilde{E}_1$. Then $w_2 > 0$ and $\tilde{E}_1 > 0$ by virtue of (3.4) and (3.5). Define in $\mathbb{R}_0^+$

$$\begin{aligned}\mathscr{G}(t) = \|u(t,\cdot)\|_2^2 &+ \int_0^t \{Q(\tau)\|u(\tau,\cdot)\|_2^2 + (\tau - t)Q'(\tau)\|u(\tau,\cdot)\|_2^2\}\,d\tau \\ &+ (T_0 - t)Q(0)\|u(0,\cdot)\|_2^2 + \beta(t+\beta_0)^2,\end{aligned}$$

where $T_0, \beta, \beta_0 > 0$ are constants which will be fixed later. Since $Q \in C^1(\mathbb{R}_0^+)$, it results

$$\begin{aligned}\mathscr{G}'(t) &= 2\langle u(t,\cdot), u_t(t,\cdot)\rangle + Q(t)\|u(t,\cdot)\|_2^2 - Q(0)\|u(0,\cdot)\|_2^2 \\ &\quad - \int_0^t Q'(\tau)\|u(\tau,\cdot)\|_2^2\,d\tau + 2\beta(t+\beta_0) \\ &= 2\langle u(t,\cdot), u_t(t,\cdot)\rangle + 2\int_0^t Q(\tau)\langle u(\tau,\cdot), u_t(\tau,\cdot)\rangle\,d\tau + 2\beta(t+\beta_0).\end{aligned}$$

From the *Distribution Identity* (A), taking $\phi = u \in K$, it follows that

$$\begin{aligned}\frac{1}{2}\mathscr{G}''(t) &= \|u_t(t,\cdot)\|_2^2 - Q(t)\langle u_t(t,\cdot), u(t,\cdot)\rangle - \langle Au(t,\cdot), u(t,\cdot)\rangle \\ &\quad + \langle f(t,\cdot,u(t,\cdot)), u(t,\cdot)\rangle + Q(t)\langle u(t,\cdot), u_t(t,\cdot)\rangle + \beta \\ &= \|u_t(t,\cdot)\|_2^2 - \langle Au(t,\cdot), u(t,\cdot)\rangle + \langle f(t,\cdot,u(t,\cdot)), u(t,\cdot)\rangle + \beta.\end{aligned}$$

Now observe that, thanks to (3.3) and $(\mathscr{F}_3)'$, we have

$$\langle Au(t,\cdot), u(t,\cdot)\rangle - \langle f(t,\cdot,u(t,\cdot)), u(t,\cdot)\rangle \leqq \gamma p_+ \mathscr{A}u(t) - q_- \mathscr{F}u(t).$$

Hence, combining these formulas with the definition (3.1) of the energy function, we get

$$\begin{aligned}\frac{1}{2}\mathscr{G}''(t) &\geqq \|u_t(t,\cdot)\|_2^2 - \gamma p_+ \mathscr{A}(u) + q_- \mathscr{F}u(t) + \beta \\ &= \left(1 + \frac{\gamma p_+}{2}\right)\|u_t(t,\cdot)\|_2^2 + (q_- - \gamma p_+)\mathscr{F}u(t) \\ &\quad - \gamma p_+ Eu(t) + \beta.\end{aligned} \tag{5.2}$$

Now $(q_- - \gamma p_+)\mathscr{F}u(t) \geqq (q_- - \gamma p_+)w_2 = \gamma p_+ \tilde{E}_1$ by (3.5), and also $Eu(t) \leqq Eu(0) - \int_0^t Q(\tau)\|u_t(\tau,\cdot)\|_2^2 \,d\tau$ by the *Energy Conservation* (B) and $(\mathscr{F}_2)$. Therefore

$$\mathscr{G}''(t) \geqq (2+\gamma p_+)\left\{\|u_t(t,\cdot)\|_2^2 + \beta\right\} + 2\gamma p_+ \int_0^t Q(\tau)\|u_t(\tau,\cdot)\|_2^2 \,d\tau, \quad (5.3)$$

where $\beta = 2\{\tilde{E}_1 - Eu(0)\} > 0$ by (3.5). Take β_0 so large that $\mathscr{G}'(0) = 2\langle u(0,\cdot), u_t(0,\cdot)\rangle + 2\beta\beta_0 > 0$. Then, since Q is non-negative in $\mathbb{R}_0^+$, it results

$$\mathscr{G}'',\ \mathscr{G}',\ \mathscr{G} > 0 \quad \text{in} \quad \mathbb{R}_0^+.$$

We assert that

$$\mathscr{G}\mathscr{G}'' - \alpha\mathscr{G}'^2 \geqq 0 \quad \text{in} \quad [0, T_0], \quad (5.4)$$

for any $T_0 > 0$, where $\alpha = (2+\gamma p_+)/4$. Put

$$\mathbb{A} = \|u(t,\cdot)\|_2^2 + \int_0^t Q(\tau)\|u(\tau,\cdot)\|_2^2 \,d\tau + \beta(t+\beta_0)^2,$$

$\mathbb{B} = \frac{1}{2}\mathscr{G}'$ and $\mathbb{C} = \|u_t(t,\cdot)\|_2^2 + \int_0^t Q(\tau)\|u_t(\tau,\cdot)\|_2^2 \,d\tau + \beta$. Since Q and $-Q'$ are non-negative in $\mathbb{R}_0^+$ by (Q), we have

$$\mathbb{A} \leqq \mathscr{G} \quad \text{in} \quad [0, T_0]. \quad (5.5)$$

Moreover, by (5.3) and the fact that $2\gamma p_+ > \gamma p_+ + 2$, being $\gamma p_+ > 2$, we get

$$\mathbb{C} \leqq \mathscr{G}''/(2+\gamma p_+) \quad \text{in} \quad \mathbb{R}_0^+. \quad (5.6)$$

Observe that for all $(\xi,\eta) \in \mathbb{R}^2$ and $t \in \mathbb{R}_0^+$

$$\begin{aligned}\mathbb{A}\xi^2 + 2\mathbb{B}\xi\eta + \mathbb{C}\eta^2 = \|\xi u(t,\cdot) + \eta u_t(t,\cdot)\|_2^2 + \int_0^t Q(\tau)\|\xi u(\tau,\cdot) \\ + \eta u_t(\tau,\cdot)\|_2^2 \,d\tau + \beta\{(t+\beta_0)\xi + \eta\}^2 \geqq 0,\end{aligned}$$

because Q is non-negative in $\mathbb{R}_0^+$. Thus $\mathbb{A}\mathbb{C} - \mathbb{B}^2 \geqq 0$. Hence, (5.4) holds by virtue of (5.5), (5.6) and the fact that $\mathbb{A}, \mathbb{C} > 0$.

Clearly $\alpha > 1$ since $\gamma p_+ > 2$ by assumption. Now (5.4) can be written as $(\mathscr{G}^{-\alpha}\mathscr{G}')' \geqq 0$, so that

$$\frac{\mathscr{G}'(t)}{\mathscr{G}^\alpha(t)} \geqq \frac{\mathscr{G}'(0)}{\mathscr{G}^\alpha(0)} > 0 \quad \text{for} \quad t \in [0, T_0].$$

This is a Riccati inequality with blow up time

$$T < \frac{1}{\alpha - 1} \cdot \frac{\mathscr{G}(0)}{\mathscr{G}'(0)}. \quad (5.7)$$

Consequently, if T_0 is chosen as the right-hand side of the above inequality we have a contradiction. In fact, since $\mathscr{G}(0)$ depends linearly on T_0, this gives an easily solved equation for T_0, the solution being positive for all β_0 large enough, for example, whenever

$$\beta\beta_0 > \frac{2}{\gamma p_+ - 2}\, Q(0)\|u(0,\cdot)\|_2^2 - \langle u(0,\cdot), u_t(0,\cdot)\rangle,$$

where $\beta = 2\{\tilde{E}_1 - Eu(0)\} > 0$. This completes the proof. □

Remark 5.1. *The next result gives new information only when $\gamma p_+ \leqq 2$ and $(\mathscr{F}_3)$ does not hold.* Otherwise it is a strict consequence of either Theorem 5.1 when $\gamma p_+ > 2$ or Theorem 3.1 if $(\mathscr{F}_3)$ holds. Indeed, for linear damping functions, satisfying (Q), condition $(\mathcal{Q}_1)$ of Section 4 is verified, with $m = 2$, $\kappa = 0$ and $\delta(t) = |\Omega|^{(q_- - 2)/q_-} Q(t)$, and so $(\mathcal{Q})$ holds. Furthermore, in Theorem 3.1 we can choose $\psi = 1$ and $k = \max\{|\Omega|^{(q_- - 2)/q_-} Q(0), 1\}$, so that (3.6) is automatic for any $\theta \in (0, \theta_0)$, where now $\theta_0 = (q_- - 2)/(q_- + 2)$.

However, Theorem 5.1 is not completely contained in Theorem 3.1, since $(\mathscr{F}_3)'$ is weaker than $(\mathscr{F}_3)$-(ii) *even when $(\mathscr{F}_3)$-(i) holds*, as the example (4.1), (4.2) shows. Indeed, any function f verifying (4.1), (4.2) satisfies $(\mathscr{F}_1)$, $(\mathscr{F}_2)$, $(\mathscr{F}_3)'$ and $(\mathscr{F}_3)$-(i), but not in general $(\mathscr{F}_3)$-(ii), when (4.3) does not hold; while the validity of (4.1)–(4.3) implies (4.4), which is exactly $(\mathscr{F}_3)'$.

We note in passing that even for linear damping the case $(\mathscr{F}_3)'$ was not covered in [23–25], while first appears in [22].

Theorem 5.2. *Suppose that $(\mathscr{M})$, (Q), $(\mathscr{F}_1)$, $(\mathscr{F}_2)$ and $(\mathscr{F}_3)'$ hold. Then there are no solutions $u \in K$ of (5.1) in $\mathbb{R}_0^+ \times \Omega$, satisfying (3.21).*

Proof. Assume by contradiction that there exists a solution $u \in K$ of problem (5.1) in $\mathbb{R}_0^+ \times \Omega$, satisfying (3.21). Define in $\mathbb{R}_0^+$ the same function $\mathscr{G}$ as in Theorem 5.1, obtaining in place of (5.2) the estimate

$$\mathscr{G}''(t) \geqq (2+q_-)\,\|u_t(t,\cdot)\|_2^2 + 2(q_- - \gamma p_+)\mathscr{A}u(t) - 2q_- Eu(t) + 2\beta.$$

Using the fact that $(q_- - \gamma p_+)\mathscr{A}u(t) \geqq (q_- - \gamma p_+)w_1 = q_- E_1$ by (3.21), from the previous relation, in place of (5.3), we get

$$\mathscr{G}''(t) \geqq (2+q_-)\left\{\|u_t(t,\cdot)\|_2^2 + \beta\right\} + 2q_- \int_0^t Q(\tau)\|u_t(\tau,\cdot)\|_2^2\, d\tau, \tag{5.8}$$

where now $\beta = 2\{E_1 - Eu(0)\} > 0$. From here on the proof is the same as that of Theorem 5.1, with q_- in place of γp_+ and $\alpha = (2+q_-)/4$. Again $\alpha > 1$, since $q_- > 2$ by (3.4). □

Remark 5.2. In all Theorems 3.1, 5.1 and 5.2, as well as in their consequences, the trivial case $Q \equiv 0$ can be included.

It is clear from the proofs of Theorems 5.1 and 5.2 that it is possible to find an upper bound for the blow up time T by virtue of (5.7). Indeed, by the choice of β_0

$$T_0 = \frac{\|u(0,\cdot)\|_2^2 + \beta\beta_0^2}{2(\alpha-1)(\beta\beta_0 - a_0)}, \quad a_0 = \frac{Q(0)\|u(0,\cdot)\|_2^2}{2(\alpha-1)} - \langle u(0,\cdot), u_t(0,\cdot)\rangle,$$

where $\alpha = (2+\gamma p_+)/4 > 1$, $\beta = 2\{\tilde{E}_1 - Eu(0)\} > 0$ in Theorem 5.1, while $\alpha = (2+q_-)/4 > 1$, $\beta = 2\{E_1 - Eu(0)\} > 0$ in Theorem 5.2. Similar results were given in [25, Theorem 3.4], when $Q \equiv 1$.

It is worth noting that *in the standard degenerate Kirchhoff case in which* $p \equiv 2$ *and* $\gamma > 1$*, the restriction* $2 < \gamma p_+$ *is automatic*. Hence Theorem 5.1 fits fairly well in the main prototype of the paper.

As a natural application of Theorems 5.1 and 5.2 and of Remark 5.1 *from now on we assume the validity of* (4.1) *and* (4.2) *on* f *and* M. As in Section 4 we distinguish two cases, depending on whether b is zero or not. Let $s = b$ and $\gamma > 1$ if $b > 0$, while $s = a$ and $\gamma = 1$ if $b = 0$. Of course we are much more interested in the *degenerate case* $a = 0$, when an effective Kirchhoff term arises in the system. Lemma 4.3 and Theorem 4.2 continue to hold and consequently (4.10)–(4.13) are still available.

Theorem 5.3. *If* $u \in K$ *is a solution of problem* (5.1) *in* $\mathbb{R}_0^+ \times \Omega$*, then* $w_2 = \inf_{t\in\mathbb{R}_0^+} \mathscr{F}u(t) > -\infty$*. If, in addition,* $Eu(0) < \tilde{E}_1$*, with* $\tilde{E}_1$ *given as in* (3.5)*, then* $w_2 > 0$*,* $\tilde{E}_1 > 0$ *and* $(\upsilon(t), Eu(t)) \in \tilde{\Sigma}$ *for all* $t \in \mathbb{R}_0^+$*, where* $\tilde{\Sigma}$ *is defined in* (4.14)*.*

There are no solutions $u \in K$ *of* (5.1) *in* $\mathbb{R}_0^+ \times \Omega$*, if either* $2 < \gamma p_+$ *and* $Eu(0) < \tilde{E}_1$*, or* $\gamma p_+ \leqq 2$ *and* $Eu(0) < E_1$*.*

Proof. The first part is a consequence of the first part of Theorem 5.1. The second part can be proved exactly as in Theorem 4.1, using (B) in place of (B)-(ii). The last part of the result is now a direct consequence of either Theorem 5.1 or Theorem 5.2. □

Corollary 5.1. *If* $u \in K$ *is a solution of* (5.1) *in* $\mathbb{R}_0^+ \times \Omega$*, with* $Eu(0) \leqq E_1$*, then* (4.18) *holds provided that* $2 < \gamma p_+$*.*

While there are no solutions $u \in K$ *of* (5.1) *in* $\mathbb{R}_0^+ \times \Omega$ *when* (4.17) *holds.*

Proof. Both statements can be proved exactly as Proposition 4.1 and Corollary 4.2 using Theorem 5.3 in place of Corollary 4.1. □

In this context condition ($\mathscr{D}$)-(ii) of Section 4 reduces simply to the request that $Q > 0$ in $[0, t_*)$.

Theorem 5.4. *If* $2 < \gamma p_+$ *and also* ($\mathscr{D}$) *holds, then problem* (5.1) *does not possess solutions* $u \in K$ *in* $\mathbb{R}_0^+ \times \Omega$*, satisfying* (4.19)*.*

Proof. The proof is exactly as for Theorem 4.3, with the use of the first part of Corollary 5.1 instead of Proposition 4.1. □

As in Section 3 we say that $u \in K$ is a *strong solution of* (5.1) if u satisfies the *Distribution Identity* (A), while (B) is replaced by the *Strong Energy Conservation* $(\mathrm{B})_{\mathrm{s}}$, that is $Eu(t) = Eu(0) - \int_0^t \{Q(\tau)\|u_t(\tau,\cdot)\|_2^2 + \mathscr{F}_t u(\tau)\}\,\mathrm{d}\tau$ for all $t \in \mathbb{R}_0^+$. We already noted in Section 3 the importance to consider (weak) solutions.

Theorem 5.5. *If* $2 \geqq \gamma p_+$ *and also* $(\mathscr{D})$ *holds, then problem* (5.1) *does not possess strong solutions* $u \in K$ *in* $\mathbb{R}_0^+ \times \Omega$*, satisfying* (4.19).

Proof. Assume by contradiction that $u \in K$ is a strong solution in $\mathbb{R}_0^+ \times \Omega$ of (5.1), satisfying (4.19). First we prove that there are no points $s > 0$ such that $\upsilon(s) = \upsilon_1$. Otherwise $Eu(s) = E_1$ and, as shown in the proof of Theorem 4.3, assumption $(\mathscr{D})$ would provide a contradiction. Hence $\upsilon(t) > \upsilon_1$ and $Eu(t) < E_1$ for all $t > 0$ by an obvious continuity argument and $(\mathrm{B})_{\mathrm{s}}$. In particular, this implies (4.15). Now fix $t_0 > 0$, so that

$$\inf_{t \in \mathbb{R}_0^+} \mathscr{A}u(t) \geqq w_1 \quad \text{and} \quad Eu(t_0) < E_1, \tag{5.9}$$

where w_1 and E_1 are given in (4.13). Put $I = [t_0, \infty)$ and define for $t \in I$,

$$\begin{aligned}\mathscr{G}(t) = \|u(t,\cdot)\|_2^2 + \int_{t_0}^t &\left\{Q(\tau)\|u(\tau,\cdot)\|_2^2 + (\tau - t)Q'(\tau)\|u(\tau,\cdot)\|_2^2\right\}\mathrm{d}\tau \\ &+ (T_0 + t_0 - t)Q(t_0)\|u(t_0,\cdot)\|_2^2 + \beta(t - t_0 + \beta_0)^2,\end{aligned}$$

where T_0, β, $\beta_0 > 0$ are constants which will be fixed later. We proceed almost exactly as in the proof of Theorem 5.2 until (5.8), with the obvious changes. Now $Eu(t) \leqq Eu(t_0) - \int_{t_0}^t Q(\tau)\|u_t(\tau,\cdot)\|_2^2\,\mathrm{d}\tau$ by the *Energy Conservation* $(\mathrm{B})_{\mathrm{s}}$ and $(\mathscr{F}_2)$, so that (5.8) is replaced by

$$\mathscr{G}''(t) \geqq (2 + q_-)\left\{\|u_t(t,\cdot)\|_2^2 + \beta\right\} + 2q_- \int_{t_0}^t Q(\tau)\|u_t(\tau,\cdot)\|_2^2\,\mathrm{d}\tau,$$

where now $\beta = 2\{E_1 - Eu(t_0)\} > 0$ by (5.9). Condition (5.4) holds in $[t_0, t_0 + T_0]$, where $\alpha = (2 + q_-)/4$ as in Theorems 5.2. But u cannot be global as in the proofs of Theorems 5.1 and 5.2, by taking $\beta_0 > 0$ so large that

$$\beta\beta_0 > \frac{2}{q_- - 2}\,Q(t_0)\|u(t_0,\cdot)\|_2^2 - \langle u(t_0,\cdot), u_t(t_0,\cdot)\rangle,$$

while

$$T_0 = \frac{4}{q_- - 2} \cdot \frac{\mathscr{G}(t_0)}{\mathscr{G}'(t_0)},$$

since again $\mathscr{G}(t_0)$ depends linearly on T_0 and the equation is solvable in T_0 thanks to the choice of β_0. Here clearly $\mathbb{A} = \|u(t,\cdot)\|_2^2 + \int_{t_0}^t Q(\tau)\|u(\tau,\cdot)\|_2^2\,\mathrm{d}\tau + \beta(t - t_0 + \beta_0)^2$, then $\mathbb{B} = \frac{1}{2}\mathscr{G}'$ and $\mathbb{C} = \|u_t(t,\cdot)\|_2^2 + \int_{t_0}^t Q(\tau)\|u_t(\tau,\cdot)\|_2^2\,\mathrm{d}\tau + \beta$. Hence (5.5) holds now in $[t_0, t_0 + T_0]$, while (5.6) is valid in I. The fact that u cannot be global shows that also this case cannot occur. □

Of course Theorem 5.5 extends and generalizes the second part of Theorem 4 of [23] and Theorem 3.1 of [24] when the damping is linear in v.

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