Periodic solutions of the Brillouin electron beam focusing equation

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Abstract

Quite unexpectedly with respect to the numerical and analytical results found in literature, we establish a new range for the existence of $2\pi-$periodic solutions of the Brillouin focusing beam equation

$$\ddot{x} + b(1 + \cos t)x = \frac{1}{x}.$$

This is possible thanks to suitable nonresonance conditions acting on the rotation number of the solutions in the phase plane.

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1 Introduction and main results

This note concerns the existence of periodic solutions of the equation

$$\ddot{x} + b(1 + \cos t)x = \frac{1}{x}, \tag{1}$$

where b is a positive constant. Throughout the paper, we will not take into account solutions with collisions, but we will always search for positive 2π-periodic solutions of (1).

The physical meaning of equation (1) arises in the context of Electronics, since it governs the motion of a magnetically focused axially symmetric electron beam under the influence of the Brillouin flow, as shown in [1]. From a mathematical point of view, (1) is a singular perturbation of a Mathieu equation, as we will explain below.

Motivated by some numerical experiments realized in [1], where it was conjectured that, if $b \in (0, 1/4)$, equation (1) should have a 2π-periodic solution, in the last fifty years the work of many mathematicians has given birth to an extensive literature about this topic. Although at the moment the conjecture has not been correctly proven yet, many advances in this line have been obtained, allowing to understand that the problem of existence of 2π-periodic solutions of (1) when $b \in (0, 1/4)$ can be really delicate, and arising doubts on the validity of the result conjectured in [1].

The first analytic work on the periodic solvability of (1) was realized by T. Ding in [7]. There, it was proven that if $b \in (0, 1/16)$, then equation (1) has at least a 2π-periodic solutions. Later, other works showed that uniqueness holds under the previous hypothesis (see for instance [14, 15, 21]).

Afterwards, Y. Ye and X. Wang [18] proved the existence of 2π-periodic solutions of (1) when $b \in (0, 2/(\pi^2 + 4)) \approx (0, 0.1442)$.

To the best of our knowledge, the following step towards the resolution of the conjecture was done by M. Zhang, extending existence to the interval $(0, 0.1532)$ using a contraction argument applied to a positive linear operator (see [19]).

A couple of years later, the same author determined in [20] the best range of b actually known for the 2π-periodic solvability of (1), using a non-resonance hypothesis for the associated Mathieu equation (a particular Hill type equation)

$$\ddot{x} + b(1 + \cos t)x = 0. \tag{2}$$

In order to prove such a result, the author considered there the function $K : [1, +\infty] \to \mathbb{R}$ defined by

$$K(\alpha) = \begin{cases} \dfrac{1}{\pi^2} & \alpha = 1 \\ \dfrac{(\alpha-1)^{1+\frac{1}{\alpha}}}{8\pi^{1-\frac{1}{2\alpha}}\alpha^{1-\frac{1}{\alpha}}(2\alpha-1)^{\frac{1}{\alpha}}} \left(\dfrac{\Gamma\left(\frac{1}{2}-\frac{1}{2\alpha}\right)}{\Gamma\left(1-\frac{1}{2\alpha}\right)}\right)^2 \left(\dfrac{\Gamma(\alpha)}{\Gamma\left(\frac{1}{2}+\alpha\right)}\right)^{\frac{1}{\alpha}} & \alpha \in (1, +\infty) \\ \dfrac{1}{8} & \alpha = +\infty; \end{cases}$$

with this definition, a sufficient condition in order for the Dirichlet problem ($x(0) = 0 = x(2\pi)$) associated with (2) to have a unique solution is that

$$b < \max_{\alpha \in [1, +\infty]} K(\alpha) \approx 0.16448.$$

Under this non-resonance condition, i.e., if $b \in (0, 0.16448)$, then (1) has at least one 2π-periodic solution. This last result has been extended to equations where the singularity may be of weak type (see [16]).

Actually, the function K is a powerful tool in order to study the existence of periodic solutions for more general versions of equation (1) (see [5, 17]).

An important result to understand the difficulty of showing the validity of the conjecture proposed in [1] was proven in [21]. In that paper, it was established an unanimous relation between the stability intervals for the Mathieu equation (2) and the existence of periodic solutions for the Yermakov-Pinney equation

$$\ddot{x} + b(1 + \cos t)x - \frac{1}{x^3} = 0. \tag{3}$$

In particular, [21, Theorem 2.1] ensures that (3) has a positive 2π-periodic solution if and only if (2) is stable in the sense of Lyapunov. Notice that the stability intervals of the Mathieu equation

$$(\lambda_0, \lambda_1'), \quad (\lambda_2', \lambda_1), \quad (\lambda_2, \lambda_3'), \quad \ldots,$$

where λ_i, $i = 0, 1, \ldots$ and λ_i', $i = 1, 2, \ldots$, respectively, are the values of the parameter b for which equation (2) has, respectively, a genuine π-periodic solution and a genuine 2π-periodic solution, are defined approximately by $\lambda_0 = 0$, $\lambda_1' \approx 1/6$; $\lambda_2' \approx 0.4$, $\lambda_1 \approx 0.95, \ldots$ (see [12, Theorem 2.1] and [4, Figure 1]). This suggests that, in order to obtain a correct proof of the conjecture by V. Bevc, J. L. Palmer and C. Süsskind, one has to take into account some property of equation (1) which is not verified by equation (3). Indeed, if we assume that (3) has at least one $2\pi-$periodic solution whenever $b \in (0, 1/4)$, then one can take b near $1/4$ in such a way that it does not belong to any stability interval of (2), and this contradicts [21, Theorem 2.1].

It has to be said that, in [13, Theorem 3.2], it was obtained that (1) has a periodic solution when $b \in (0, 1/4)$. Unfortunately, such a theorem seems applicable to equation (3), as well, obtaining the same conclusion, but this it is not possible according to the previous discussion. This contradiction seems to leave the conjecture in [1] still open.

However, in this work we are not able to prove or disprove the result which was conjectured in [1], but we will show that (1) may have periodic solutions also when b belongs to intervals other than $(0, 1/4)$. In fact, we will prove the following.

Theorem 1 *If* $b \in [0.4705, 0.59165]$*, then* (1) *has at least one* 2π*-periodic solution.*

This result seems in some sense unexpected, according to the numerical ones obtained in [1], where it was observed that when the parameter b begins to cross the umbral $1/4$, the 2π-periodic solvability of (1) is not clear. It seems indeed that Theorem 1 is the first result of existence for (1) when b does not belong to the first stability interval of equation (2) (notice that we are dealing with values of the parameter b belonging to the second stability interval of equation (2), of course agreeing with [21, Theorem 2.1]). Moreover, as it can be seen in Remark 3, our result is in some sense optimal, when some additional control is required on the nonlinearity.

Theorem 1 follows from a general existence result, Theorem 2 below, obtained thanks to suitable nonresonance assumptions which can be traced back to the work [8] by Fabry, as explained in Remark 2. The main abstract tool to obtain such a statement is embodied by the Poincaré-Bohl fixed point theorem.

The structure of this short note is as follows: in Section 2, we will prove Theorem 2 concerning strongly singular perturbations of a Mathieu equation. As a consequence, in Section 3 we will prove a general proposition (Proposition 1) allowing to prove Theorem 1.

2 A non-resonance theorem for singular perturbations of a Mathieu equation

The proof of Theorem 1 is based on a non-resonance result which involves nonlinearities with "atypical" linear growth, and could have interest by itself.

Theorem 2 *Let us assume that there exist positive constants A_+, B_+ such that*

$$\frac{1}{2\pi}\int_0^{2\pi} \min\left\{\frac{b(1+\cos t)}{B_+}, 1\right\} dt \; > \frac{n}{2\sqrt{B_+}}, \tag{4}$$

$$\frac{1}{2\pi}\int_0^{2\pi} \max\left\{\frac{b(1+\cos t)}{A_+}, 1\right\} dt \; < \frac{n+1}{2\sqrt{A_+}}, \tag{5}$$

for some natural number n. Then (1) *has at least one 2π-periodic solution.*

Before introducing the main tools to prove the theorem, a couple of remarks are in order.

Remark 1 With the aim of keeping the exposition at a rather simple level, and taking into account that our main goal will be to study the existence of 2π-periodic solutions of (1), we will always consider equation (1) as a starting point. However, the result can be extended, with the same approach and similar computations, to more general equations like

$$\ddot{x} + a(t)x + g(t,x) = 0, \tag{6}$$

where $a(t)$ is continuous and 2π-periodic, and $g : [0, 2\pi] \times (0, +\infty) \to \mathbb{R}$ has a similar behavior as $-1/x^\gamma$, with $\gamma \geq 1$, near $x = 0$, being allowed to grow at most sublinearly at infinity. For instance, as in [11], one can assume that there exist $\sigma > 0$ and a continuous function $f : (0, \sigma] \to \mathbb{R}$ such that $g(t,x) \leq f(x)$, whenever $x \in (0, \sigma]$, and

$$\lim_{r\to 0^+} f(r) = -\infty, \qquad \int_0^\sigma f(r)\,dr = -\infty.$$

Of course, in this case $a(t)$ will replace $b(1 + \cos t)$.

Remark 2 Conditions (4) and (5) were introduced by Fabry in [8] for the equation

$$\ddot{x} + g(t,x) = 0,$$

with

$$p(t) \leq \liminf_{|x|\to+\infty} \frac{g(t,x)}{x} \leq \limsup_{|x|\to+\infty} \frac{g(t,x)}{x} \leq q(t),$$

asking that

$$\sqrt{\lambda_j} < \sup_{\xi>0} \frac{\frac{1}{2\pi}\int_0^{2\pi} \min\{p(t), \xi\}\,dt}{\sqrt{\xi}}, \qquad \inf_{\xi>0} \frac{\frac{1}{2\pi}\int_0^{2\pi} \max\{q(t), \xi\}\,dt}{\sqrt{\xi}} < \sqrt{\lambda_{j+1}},$$

where λ_j is the j-th eigenvalue of the considered 2π-periodic problem. Such conditions are usually coupled with the sign assumption

$$\liminf_{|x|\to+\infty} \operatorname{sgn} x\, f(t,x) > 0$$

(see for instance [9]), which, however, in the model case $g(t,x) = b(1+\cos t)x + f(t,x)$, with $\lim_{|x|\to+\infty} f(t,x) = 0$, is not satisfied. This is one of the main difficulties of the problem considered in the present paper.
As it is easy to see, (4) and (5) are the counterpart of Fabry's conditions for the Dirichlet spectrum (which is the natural one to consider when dealing with problems with a singularity, see [20]).

Remark 3 As a consequence of Theorem 2, we can obtain the main results in [6, 11]. Indeed, assume that there exist *positive* constants A_+, B_+ such that

$$B_+ \le a(t) \le A_+ \qquad \text{for every } t \in [0, 2\pi]. \tag{7}$$

Then, according to [6, 11], there exists at least one 2π-periodic solution of (6) under the nonresonance assumption

$$\left(\frac{n}{2}\right)^2 < B_+ \le A_+ < \left(\frac{n+1}{2}\right)^2,$$

where $n \in \mathbb{N}$. It is easy to obtain this result from Theorem 2, since from (7) we deduce that

$$\frac{a(t)}{A_+} \le 1 \le \frac{a(t)}{B_+} \qquad \text{for every } t \in [0, 2\pi]. \tag{8}$$

Under (7), from the point of view of resonance, the results in [6, 11] are optimal, in view of the counterexample produced in [2], according to which there exist forcing terms $e(t)$ such that the equation

$$\ddot{x} - \frac{1}{x^3} + \frac{1}{4}x = e(t)$$

has no 2π-periodic solutions. Thus, Theorem 2 seems to be optimal whenever we are able to control $a(t)$ with estimates like (7) and (8), essentially requiring, in this case, a nonresonance assumption. On the other hand, the mean conditions (4) and (5) do not ask that $a(t)$ is controlled like in (7) and (8), allowing it to possibly cross some eigenvalues (cf. [3]) as in our case, being $0 \le a(t) \le 2b$.

We are now going to prove Theorem 2. As it was mentioned previously, we will have to overcome the difficulty of working with nonlinearities with atypical linear growth, since, in our concrete case, the nonlinearity grows linearly towards the function $b(1+\cos t)x$, which vanishes at some times. For this reason, classical arguments in literature (like the ones in [9, 10, 11]) do not extend as they are to (1), because it is not possible to construct an admissible spiral which allows to control the dynamics of the solutions.

We will prove Theorem 2 by means of some preliminary lemmas. To this aim, it will be convenient to introduce the "norm" application defined as

$$\mathcal{N} : \Lambda \to \mathbb{R}, \qquad \mathcal{N}(x,y) := bx^2 + y^2 - 2\ln x,$$

being Λ the half-plane with positive abscissa, i.e. $\Lambda = \{(x,y) \in \mathbb{R}^2 : x > 0\}$. Fixed a value c of the function $\mathcal{N}(x,y)$, we will denote the corresponding level curve by γ_c, i.e.,

$$\gamma_c = \{(x,y) \in \Lambda : \mathcal{N}(x,y) = c\}.$$

It is worth observing that the function $\mathcal{N}(x,y)$ reaches its minimum in the point $P_0 = (1/\sqrt{b}, 0)$, where it takes the value $1 - 2\ln(1/\sqrt{b})$ (possibly negative for some values of the parameter b). For values of the energy greater than $1 - 2\ln(1/\sqrt{b})$, the level curves of $\mathcal{N}$ turn around P_0, being the union of two symmetric arcs joining on the x-axis.

We will look at the solutions of (1) in the phase plane, taking thus into account the couple $(x, \dot{x})$. As already mentioned, we are interested in positive solutions, so that we will take into account the dynamics of the solutions in the right half-plane.

The first lemma ensures the global continuability of the solutions, i.e., shows that the maximal domain of every solution of (1) is $[0, +\infty)$.

Lemma 1 *Let $x(t)$ be a solution of* (1) *(not necessarily periodic). Then*

$$\mathcal{N}(x(t), \dot{x}(t)) < +\infty \qquad \textit{for every } t \geq 0.$$

Proof. Since

$$\lim_{x \to +\infty} \frac{bx^2}{bx^2 - 2\ln x} = 1,$$

taking $C > \max\{1, b\}$ there exists $K_0 > 1$ such that

$$\frac{b}{2}(x^2 + y^2) \leq \frac{C}{2}(bx^2 + y^2 - 2\ln x) \qquad \text{for every } x \geq K_0, \quad y \in \mathbb{R}. \tag{9}$$

For every solution $x(t)$ of (1), we define the function

$$U : I \to \mathbb{R}, \qquad U(t) = \frac{\mathcal{N}(x(t), \dot{x}(t))}{2} = \frac{1}{2}(bx(t)^2 + \dot{x}(t)^2 - 2\ln x(t)),$$

where I is the maximal domain of $x(t)$. We are going to prove that $I = [0, +\infty)$.
Since

$$U'(t) = -bx(t)\dot{x}(t)\cos t,$$

for $t \in I$ we have that

$$U'(t) \leq \frac{b(x(t)^2 + \dot{x}(t)^2)}{2},$$

from which it can be deduced that

$$U'(t) \leq CU(t) + C\ln K_0 \qquad \text{for every } t \in I. \tag{10}$$

Indeed, if $t \geq 0$ is such that $x(t) \geq K_0$, then (9) implies that $U'(t) \leq CU(t)$. On the contrary, if $x(t) \leq K_0$, we deduce that either $x(t) \leq 1$, and then $U'(t) \leq CU(t)$, or $1 \leq x(t) \leq K_0$, and thus (10) holds. Now, according to the Gronwall-Bellman Lemma, the result is proven. ∎

As it was mentioned in the previous discussion, equations like (1) do not admit the existence of an admissible spiral controlling the solutions. However, the following result ensures that (1) has the "property of elasticity", at least locally. Roughly speaking, this means that if there is a time when the norm of the solution is large enough, then, for every preceding time instant, the solution had to be large (in norm). Precisely, we have the following.

Lemma 2 *Let $\rho_0 > 0$ be sufficiently large. Then, there exists $R_1 > \rho_0$ such that, for every solution $x(t)$ of (1) satisfying*

$$\mathcal{N}(x(t_1), \dot{x}(t_1)) \geq R_1$$

for some $t_1 > 0$, it holds

$$\mathcal{N}(x(t), \dot{x}(t)) \geq \rho_0 \qquad \text{for every } t \in [0, t_1].$$

Proof. We first observe that there exists a constant $M > 0$ such that

$$\frac{b(x^2 + y^2)}{\mathcal{N}(x, y)} < M, \tag{11}$$

for every $(x, y) \in \Lambda$. Now, choosing $\rho_0 > 0$ sufficiently large, there exist $u_0^- < 1 < u_0^+$ such that

$$\gamma_{\rho_0} = \text{Graph}(F_0) \cup \text{Graph}(-F_0),$$

where $F_0 : (0, +\infty) \to \mathbb{R}$ is a function such that $F_0(u_0^-) = F_0(u_0^+) = 0$, having constant sign on (u_0^-, u_0^+).
Let us fix L_1 satisfying

$$2L_1 \geq \max_{x \in [u_0^-, u_0^+]} bx^2 - 2 \ln x + 2\rho_0,$$

and consider the set of the couples $(x, y) \in \gamma_{2L_1}$: explicitly,

$$\gamma_{2L_1} = \left\{ (x, y) \in \Lambda : y = \pm\sqrt{2L_1 - (bx^2 - 2 \ln x)} \right\}.$$

Thus, there exist $u_1^- < u_0^- < u_0^+ < u_1^+$ such that, similarly as before,

$$\gamma_{2L_1} = \text{Graph}(F_1) \cup \text{Graph}(-F_1),$$

where $F_1 : (0, +\infty) \to \mathbb{R}$ is defined by $F_1(x) = \sqrt{2L_1 - (bx^2 - 2 \ln x)}$ (and consequently vanishes in u_1^-, u_1^+). On the other hand, we take $L_2 > \mathrm{e}^{2\pi M} L_1$, and consider the level curve γ_{2L_2}, which is explicitly given by

$$\gamma_{2L_2} = \left\{ (x, y) \in \Lambda : y = \pm\sqrt{2L_2 - (bx^2 - 2 \ln x)} \right\}.$$

Finally, we fix $R_1 > 2L_2$, so that

$$\gamma_{2L_2} \subset \{(x, y) \in \Lambda : \mathcal{N}(x, y) \leq R_1\}.$$

Assume that there exists $x(t)$ solving (1) such that $\mathcal{N}(x(t_1), \dot{x}(t_1)) \geq R_1$, but there is $t_* \in [0, t_1)$ such that $\mathcal{N}(x(t_*), \dot{x}(t_*)) \leq \rho_0$. By continuity, we can assume that there exist $t_* < t^*$ such that $(x(t_*), \dot{x}(t_*)) \in \gamma_{2L_1}$ and $(x(t^*), \dot{x}(t^*)) \in \gamma_{2L_2}$; setting, as in Lemma 1, $U(t) = \mathcal{N}(x(t), \dot{x}(t))/2$, this explicitly means that

$$L_1 < U(t) < L_2 \qquad \text{for every } t \in (t_*, t^*), \qquad U(t_*) = L_1, \quad U(t^*) = L_2. \tag{12}$$

According to (11) and (12), from the definition of $U(t)$ we deduce that

$$U'(t) \leq MU(t), \qquad \text{for every } t \in [t_*, t^*],$$

which implies, thanks to the Gronwall-Bellman Lemma, that

$$U(t) \leq \mathrm{e}^{2\pi M} L_1 \qquad \text{for every } t \in [t_*, t^*].$$

This, however, contradicts (12), in view of the definition of L_2. ■

Now, intuitively speaking, we will prove that either the solutions of (1) have the global elasticity property, or their norm in the instant $t = 2\pi$ is lower than in the initial one. This property is useful, and it is similar to the one introduced in [10].

Lemma 3 *Let $\rho_0 > 0$ be sufficiently large. Then, there exists $R_2 > \rho_0$ such that, for every solution $x(t)$ of (1) fulfilling*

$$\max_{t \in [0, 2\pi]} \mathcal{N}(x(t), \dot{x}(t)) \geq R_2, \tag{13}$$

it is either

$$\mathcal{N}(x(t), \dot{x}(t)) \geq \rho_0 \qquad \textit{for every } t \in [0, 2\pi], \tag{14}$$

or

$$\mathcal{N}(x(0), \dot{x}(0)) > \mathcal{N}(x(2\pi), \dot{x}(2\pi)). \tag{15}$$

Proof. Let us take R_1 as in the statement of Lemma 2, for the fixed ρ_0. In the same way, we apply again Lemma 2, this time with R_1 playing the role of ρ_0, finding the corresponding R_2 for which the statement holds.
Assume now that there exists a solution $x(t)$ of (1) satisfying (13), for which it is

$$\mathcal{N}(x(0), \dot{x}(0)) \leq \mathcal{N}(x(2\pi), \dot{x}(2\pi)). \tag{16}$$

Since there exists $t_2 \in [0, 2\pi]$ such that $\mathcal{N}(x(t_2), \dot{x}(t_2)) \geq R_2$, Lemma 2 implies that $\mathcal{N}(x(0), \dot{x}(0)) \geq R_1$, so that, in view of (16), $\mathcal{N}(x(2\pi), \dot{x}(2\pi)) \geq R_1$. Consequently, using again Lemma 2, we obtain that $\mathcal{N}(x(t), \dot{x}(t)) \geq \rho_0$ for $t \in [0, 2\pi]$. ■

We are now able to show that an adaptation of the arguments in [9, 11] to our equation allows to prove that the global elasticity property cannot be fulfilled for solutions of (1) with large norm which perform an integer number of revolutions when t goes from 0 to 2π.

Lemma 4 *Under the hypotheses of Theorem 2, there exists $R_2 > 0$ such that, if $x(t)$ is a solution of* (1) *which satisfies*

$$\max_{t\in[0,2\pi]} \mathcal{N}(x(t),\dot{x}(t)) \geq R_2$$

and $(x(t),\dot{x}(t))$ performs an integer number of turns around $(1,0)$ in the time interval $[0,2\pi]$, then (15) *holds.*

Proof. In view of (4), (5), there exists a positive number δ such that

$$\frac{1}{2\pi}\int_0^{2\pi} \min\left\{\frac{b(1+\cos t)-\delta}{B_+},1\right\} dt > \frac{n}{2\sqrt{B_+}},$$
$$\frac{1}{2\pi}\int_0^{2\pi} \max\left\{\frac{b(1+\cos t)+\delta}{A_+},1\right\} dt < \frac{n+1}{2\sqrt{A_+}}.$$

In correspondence of δ, we can find $K_\delta > 0$ such that, for every $x \in [1,+\infty)$ and every $t \geq 0$,

$$\begin{aligned}[][b(1+\cos t)-\delta)](x-1)^2 - K_\delta &< \left[b(1+\cos t)x - \frac{1}{x}\right](x-1) \\ &< [b(1+\cos t)+\delta)](x-1)^2 + K_\delta.\end{aligned} \tag{17}$$

Moreover, we choose ρ_1 and B'_+ large, in such a way that the following relations hold:

$$\left(\frac{1}{\sqrt{B_+}}+\frac{1}{\sqrt{B'_+}}\right)^{-1}\left[\frac{1}{2\pi}\int_0^{2\pi} \min\left\{\frac{b(1+\cos t)-\delta}{B_+},1\right\} dt - \frac{K_\delta}{\rho_1}\right] > \frac{n}{2}, \tag{18}$$

$$\sqrt{A_+}\left[\frac{1}{2\pi}\int_0^{2\pi} \max\left\{\frac{b(1+\cos t)+\delta}{A_+},1\right\} dt + \frac{K_\delta}{\rho_1}\right] < \frac{n+1}{2}. \tag{19}$$

In order to perform the estimates leading to the result, we first fix $\rho_0 > 0$ sufficiently large and apply Lemma 3 in order to find $R_2 > \rho_0$ such that the statement therein holds. Then, we fix a solution $x(t)$ of (1) satisfying (13) and such that, in the phase plane, the couple $(x(t),\dot{x}(t))$ performs an integer number of revolutions around $(1,0)$ - say $k \in \mathbb{N}$ - during the time interval $[0,2\pi]$.
Thus, assume by contradiction that (15) is not satisfied; then, in view of Lemma 3, $x(t)$ fulfills (14). We are now going to estimate the time needed by $(x(t),\dot{x}(t))$ to rotate k times around the point $(1,0)$, by dividing the half-plane Λ in vertical strips and analyzing the behavior of the solution in each strip, following the procedure used in [11].
As a first step, we perform our estimates in the strip $\{x > 1\}$. Passing to modified polar coordinates around $(1,0)$ by writing

$$-\mu x = -\mu + \rho\sin\vartheta, \qquad \dot{x} = \rho\cos\vartheta,$$

where $\mu > 0$, we obtain

$$-\dot{\vartheta}(t) = \mu\frac{\dot{x}(t)^2 - \ddot{x}(t)(x(t)-1)}{\mu^2(x(t)-1)^2+\dot{x}(t)^2} \quad \text{for every } t \in [0,2\pi]. \tag{20}$$

Setting

$$J_+ = \{t \in [0, 2\pi] : x(t) \geq 1\}, \qquad J_- = \{t \in [0, 2\pi] : x(t) < 1\},$$

in view of the properties of the modified rotation numbers (see, for instance, [8]) we have that

$$2\pi \cdot \frac{k}{2} = -\int_{J_+} \dot{\theta}(t)\, dt.$$

Consequently, in view of (17),

$$\begin{aligned}
\frac{k}{2} &\geq \frac{\mu}{2\pi} \int_{J_+} \frac{\dot{x}^2 + [b(1+\cos t) - \delta](x-1)^2}{\mu^2(x-1)^2 + \dot{x}^2} dt - \frac{\mu}{2\pi} \int_{J_+} \frac{K_\delta}{\mu^2(x-1)^2 + \dot{x}^2} dt \\
&\geq \frac{\mu}{2\pi} \int_{J_+} \frac{\min\left\{\frac{b(1+\cos t)-\delta}{\mu^2}, 1\right\}(x-1)^2 + (\dot{x}/\mu)^2}{(x-1)^2 + (\dot{x}/\mu)^2} dt - \frac{\mu}{2\pi} \int_{J_+} \frac{K_\delta}{\mu^2(x-1)^2 + \dot{x}^2} dt.
\end{aligned}$$

Taking into account that the function

$$\Psi : [0, +\infty) \to \mathbb{R}, \qquad \Psi(y) = \frac{\alpha + y}{\beta + y} \tag{21}$$

is non-decreasing for $\alpha \leq \beta$, choosing $\mu = \sqrt{B_+}$, $\alpha = \min\left\{\frac{b(1+\cos t)-\delta}{\mu^2}, 1\right\}(x-1)^2$, $\beta = (x-1)^2$ and $y = (\dot{x}/\mu)^2$ we have

$$\frac{k}{2} \geq \frac{\sqrt{B_+}}{2\pi} \int_{J_+} \min\left\{\frac{b(1+\cos t) - \delta}{B_+}, 1\right\} dt - \frac{\sqrt{B_+}}{2\pi} \int_{J_+} \frac{K_\delta}{B_+(x-1)^2 + \dot{x}^2} dt. \tag{22}$$

Without loss of generality, we can assume (up to enlarging ρ_0) that R_2 is sufficiently large, so that

$$B_+(x-1)^2 + \dot{x}^2 \geq \rho_1, \qquad \text{for every } t \in J_+.$$

Therefore, (22) implies

$$\frac{k}{2\sqrt{B_+}} \geq \frac{1}{2\pi} \int_{J_+} \min\left\{\frac{b(1+\cos t) - \delta}{B_+}, 1\right\} dt - \frac{K_\delta}{\rho_1}. \tag{23}$$

We now pass to compute the time spent by $(x(t), \dot{x}(t))$ to perform $k/2$ revolutions on the "left" half phase plane, i.e. when $x \in (0, 1]$. Preliminarily, we fix

$$\widetilde{\eta} < \frac{2\pi}{\sqrt{B'_+}}, \qquad K = \left(\frac{2\pi}{\widetilde{\eta}}\right)^2 \tag{24}$$

and observe that, since

$$\lim_{x \to 0^+} b(1+\cos t)x - \frac{1}{x} = -\infty,$$

there exists $0 < d < 1$ such that

$$b(1+\cos t)x - \frac{1}{x} < -K \quad \text{ for every } x \in (0, d]. \tag{25}$$

In this way it is possible to define both the sets

$$J_d^- = \{t \in J_- \mid x(t) \le d\}\,, \qquad J_d^+ = \{t \in J_- \mid d < x(t) < 1\}$$

and, correspondingly, the time instants t_1, t_2, t_3 and t_4 (as in Figure 1) such that, in the time $t_4 - t_1$, the couple $(x(t), \dot{x}(t))$ performs half a turn in the "left" half phase plane ($x \in (0, 1]$), and

$$x(t_1) = 1 = x(t_4), \quad x(t_2) = d = x(t_3), \quad (t_1, t_2) \cup (t_3, t_4) \subseteq J_d^+, \qquad [t_2, t_3] \subseteq J_d^-\,.$$

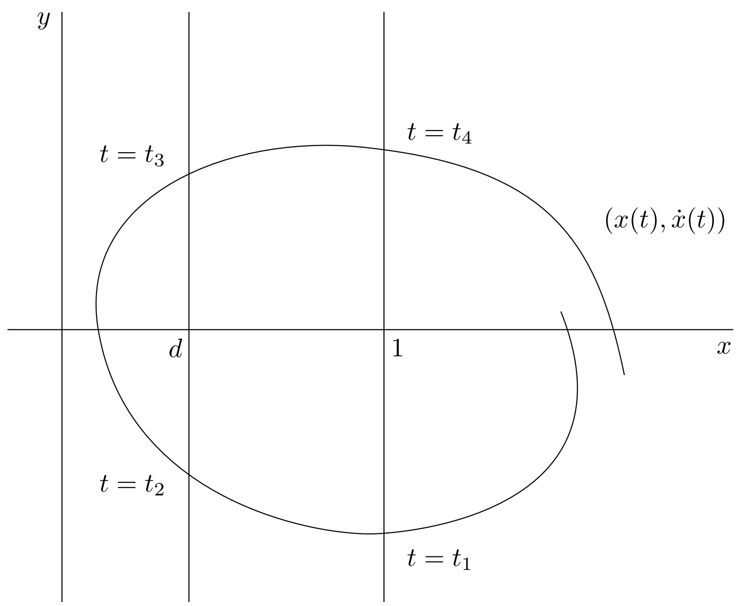

Figure 1: Defining the times t_1, t_2, t_3, t_4.

Passing to usual polar coordinates around $(1, 0)$, i.e.,

$$x = 1 + \rho\cos\vartheta, \qquad \dot{x} = \rho\sin\vartheta,$$

we arrive at

$$-\,\dot{\vartheta}(t) = \frac{\dot{x}(t)^2 - \ddot{x}(t)(x(t) - 1)}{(x(t) - 1)^2 + \dot{x}(t)^2}. \tag{26}$$

In view of (25), we deduce that

$$-\dot{\vartheta}(t) > K\cos^2\vartheta(t) + \sin^2\vartheta(t), \quad t \in [t_2, t_3],$$

so that

$$\begin{aligned} t_3 - t_2 &= \int_{\vartheta(t_3)}^{\vartheta(t_2)} \frac{ds}{K\cos^2 s + \sin^2 s} ds \\ &= \frac{1}{\sqrt{K}} \left[\arctan\left(\frac{\tan\vartheta(t_2)}{K} \right) - \arctan\left(\frac{\tan\vartheta(t_3)}{K} \right) \right] \\ &\le \frac{\pi}{2\sqrt{K}}. \end{aligned}$$

According to (24), it follows that $t_3 - t_2 < \widetilde{\eta}/2$; repeating the argument for every revolution made by $(x, \dot{x})$ around $(1, 0)$ yields

$$\operatorname{meas}\left(J_d^-\right) < \frac{k}{4}\widetilde{\eta}. \tag{27}$$

In order to compute $t_2 - t_1$, we observe that, thanks to (26), it holds

$$-\dot{\vartheta}(t) \ge \frac{\dot{x}(t)^2 - \widetilde{C}|1-d|}{(1-d)^2 + \dot{x}(t)^2} \qquad \text{for every } t \in [t_1, t_2],$$

where $\widetilde{C} = \max_{x\in[d,1]} 2bx + 1/x$. Again, we assume that ρ_0 is large enough, so that $-\dot{\vartheta}(t) > 1/2$ on $[t_1, t_2]$, and $t_2 - t_1 < \widetilde{\eta}/4$. Analogously, one can prove that $t_4 - t_3 < \widetilde{\eta}/4$, having thus that

$$\operatorname{meas}(J_d^+) < \frac{k}{2}\frac{\widetilde{\eta}}{2}.$$

Thus, in view of (24) and (27), we deduce that

$$\operatorname{meas}(J_-) = \operatorname{meas}(J_d^+) + \operatorname{meas}(J_d^-) < \frac{k}{2}\widetilde{\eta} < k\,\frac{\pi}{\sqrt{B'_+}},$$

from which

$$\frac{k}{2\sqrt{B'_+}} > \frac{1}{2\pi}\operatorname{meas}(J_-).$$

Summing up, from (23) we have

$$\frac{k}{2}\left(\frac{1}{\sqrt{B_+}} + \frac{1}{\sqrt{B'_+}}\right) \ge \frac{1}{2\pi}\int_0^{2\pi} \min\left\{\frac{b(1+\cos t) - \delta}{B_+}, 1\right\} dt - \frac{K_\delta}{\rho_1}. \tag{28}$$

On the other hand, reasoning on (20) with a similar argument and taking (17) into account, we have

$$\frac{k}{2} \le \frac{\mu}{2\pi}\int_{J_+} \frac{\max\{\frac{b(1+\cos t)+\delta}{A_+}, 1\}(x-1)^2 + (\dot{x}/\mu)^2}{(x-1)^2 + (\dot{x}/\mu)^2} dt + \frac{\mu}{2\pi}\int_{J_+} \frac{K_\delta}{\mu^2(x-1)^2 + \dot{x}^2} dt.$$

Since the function Ψ defined in (21) is non-increasing whenever $\alpha \ge \beta$, choosing $\mu = \sqrt{A_+}$ and taking $\alpha = \max\left\{\frac{b(1+\cos t)+\delta}{A_+}, 1\right\}(x-1)^2$ and $\beta = (x-1)^2$, we obtain

$$\frac{k}{2} \le \frac{\sqrt{A_+}}{2\pi}\int_{J_+} \max\{\frac{b(1+\cos t)+\delta}{A_+}, 1\} dt + \frac{\sqrt{A_+}}{2\pi}\int_{J_+} \frac{K_\delta}{A_+(x-1)^2 + \dot{x}^2} dt.$$

Again, we can assume ρ_0 (and thus R_2) so large that

$$\sqrt{A_+}(x(t)-1)^2 + \dot{x}(t)^2 \geq \rho_1, \quad t \in J_+.$$

Hence,

$$\frac{k}{2\sqrt{A_+}} \leq \frac{1}{2\pi}\int_{J_+} \max\left\{\frac{b(1+\cos t)+\delta}{A_+}, 1\right\} dt + \frac{K_\delta}{\rho_1}. \tag{29}$$

We are now able to conclude the proof. Assume first that $x(t)-1$ has at most $2n$ zeros. Then $k \leq n$, but this contradicts (18) and (28). On the contrary, if $x(t)-1$ has at least $2n+2$ zeros (notice that, in any case, the number of zeros is even), since $k \in \mathbb{N}$ it has to be $k \geq n+1$. However, this contradicts (19) and (29). The proof is completed. ■

Remark 4 In [3], the relationships between conditions (4) and (5) and the rotation number of "large" solutions of a first order planar system were highlighted. This perfectly agrees with what we have seen in the proof which has just been performed; indeed, conditions (4) and (5) force the solutions of the Cauchy problems associated with (1) not to perform an integer number of turns around $(1,0)$ in the time interval $[0,2\pi]$. Thus, they turn to be hypotheses on the number of rotations made by the solutions of equation (1) in the phase plane.

Using the previous results, a basic application of the Poincaré-Bohl Theorem allows to prove Theorem 2.

Proof of Theorem 2. Let us take R_2 sufficiently large satisfying Lemma 4 and set $B = \{(x,y) \in \Lambda : \mathcal{N}(x,y) \leq R_2\}$. In view of Lemma 1, the Poincaré map

$$P : B \to \mathbb{R}^2, \qquad P(x_0, y_0) = (x(2\pi), \dot{x}(2\pi)),$$

where $(x(t), \dot{x}(t))$ is the unique solution of the problem

$$\ddot{x} + b(1+\cos t)x - \frac{1}{x} = 0, \qquad x(0) = x_0 > 0, \quad \dot{x}(0) = y_0,$$

is well defined. Moreover, if (x_0, y_0) is a fixed point of P, then it is $(x(0), \dot{x}(0)) = (x(2\pi), \dot{x}(2\pi))$, i.e., $x(t)$ is a 2π-periodic solution of (1). Therefore, to get the conclusion it is sufficient to prove that P has a fixed point. However, if we denote by τ_1 (resp. τ_{-1}) the unitary right (resp. left) translation in the plane $(x, \dot{x})$, the map $\Phi := \tau_{-1} \circ P \circ \tau_1 : \tau_{-1}(B) \to \mathbb{R}^2$ satisfies all the hypotheses of the Poincaré-Bohl fixed point theorem, since $0 \in \tau_{-1}(B)$ and $\Phi(u) \neq \lambda u$ for every $\lambda > 1$ and every $u \in \partial\tau_{-1}(B) = \tau_{-1}\partial B$, in view of Lemma 4. Consequently, denoting by $\bar{x}$ such a fixed point, $\tau_1\bar{x}$ is a fixed point for P and the statement is proved. ■

3 Proof of Theorem 1

In order to prove Theorem 1 it will be convenient, for any $n \in \mathbb{N}$, to define the absolutely continuous functions $F_n, G_n : (0,+\infty) \times (0,+\infty) \to \mathbb{R}$ by

$$\begin{aligned} F_n(b,x) &= \frac{1}{2\pi}\int_0^{2\pi} \min\left\{\frac{b(1+\cos t)}{\sqrt{x}}, \sqrt{x}\right\} dt - \frac{n}{2}, \\ G_n(b,x) &= \frac{1}{2\pi}\int_0^{2\pi} \max\left\{\frac{b(1+\cos t)}{\sqrt{x}}, \sqrt{x}\right\} dt - \frac{n+1}{2}. \end{aligned}$$

Such functions are non-decreasing with respect to the variable b. Moreover, if there exists $n \in \mathbb{N}$ such that $\inf_{x>0} G_n(b,x) < 0$ and $\sup_{x>0} F_n(b,x) > 0$, then Theorem 2 implies that (1) has at least one 2π-periodic solution. Therefore, we have the following proposition.

Proposition 1 *Assume that there exists $n \in \mathbb{N}$ such that*

$$b \in \left(\inf\left\{b>0 : \sup_{x>0} F_n(b,x) > 0\right\}, \sup\left\{b>0 : \inf_{x>0} G_n(b,x) < 0\right\}\right). \tag{30}$$

Then, (1) has at least one 2π-periodic solution.

Let us first observe that, in view of the continuity and the monotonicity of the functions F_n, G_n in the variable b, there exist b_0^n and b_1^n such that

$$\left\{b>0 : \sup_{x>0} F_n(b,x) > 0\right\} = (b_0^n, +\infty),$$

and

$$\left\{b>0 : \inf_{x>0} G_n(b,x) < 0\right\} = (0, b_1^n).$$

The point is to prove that these two intervals contain common points, i.e., $b_0^n < b_1^n$. We will show this in the case when $n = 0$ and $n = 1$, and the estimates performed in this last case will allow to achieve the new result consisting in Theorem 1.
In particular, a gross estimation of the interval in (30) would lead to prove existence for

$$b \in \left(\frac{n^2}{2}, \frac{(n+1)^2}{4}\left(\frac{\pi}{1+\pi}\right)^2\right). \tag{31}$$

Indeed, setting $B_+ = 2b$, since $b > n^2/2$ we have

$$F_n(b, B_+) = \frac{1}{2\pi}\sqrt{b}\int_0^{2\pi} \min\left\{\frac{1+\cos t}{\sqrt{2}}, \sqrt{2}\right\} dt - \frac{n}{2} = \sqrt{\frac{b}{2}} - \frac{n}{2} > 0.$$

On the other hand, we choose

$$A_+ = \frac{4b^2}{(n+1)^2}\left(\frac{\pi+1}{\pi}\right)^2,$$

so that, since $b < \frac{1}{4}(n+1)^2(\pi/(1+\pi))^2$,

$$G_n(b, A_+) = \frac{1}{2\pi}\int_0^{2\pi} \max\left\{\frac{(n+1)\pi(1+\cos t)}{2(\pi+1)}, \frac{2b}{n+1}\frac{\pi+1}{\pi}\right\} dt - \frac{n+1}{2}$$
$$< \frac{1}{2\pi}\frac{(n+1)\pi}{2(\pi+1)}\int_0^{2\pi} \max\{1+\cos t, 1\} dt - \frac{n+1}{2} = 0.$$

Now, in order for the interval in (31) to be nonempty, we need

$$\frac{n^2}{2} < \frac{(n+1)^2}{4}\left(\frac{\pi}{1+\pi}\right)^2,$$

which approximately requires $n < 1.1$. Since $n \in \mathbb{N}$, we can take either $n = 0$ or $n = 1$, so that the 2π-periodic solvability of (1) is guaranteed whenever

$$b \in \left(0, \frac{1}{4}\left(\frac{\pi}{1+\pi}\right)^2\right) \cup \left(\frac{1}{2}, \left(\frac{\pi}{1+\pi}\right)^2\right).$$

However, taking into account that F_1, G_1 are non-decreasing, we can use a numerical approach to estimate the interval in (30) and try to compute approximately, by means of a numerical software, its endpoints, obtaining

$$\sup_{x>0} F_1(0.4705, x) > 0 \qquad (\text{but } \sup_{x>0} F_1(0.47, x) < 0)$$

and

$$\inf_{x>0} G_1(0.59165, x) < 0 \qquad (\text{but } \inf_{x>0} G_1(0.592, x) > 0)$$

(see Figures 2 and 3), whence the statement of Theorem 1.

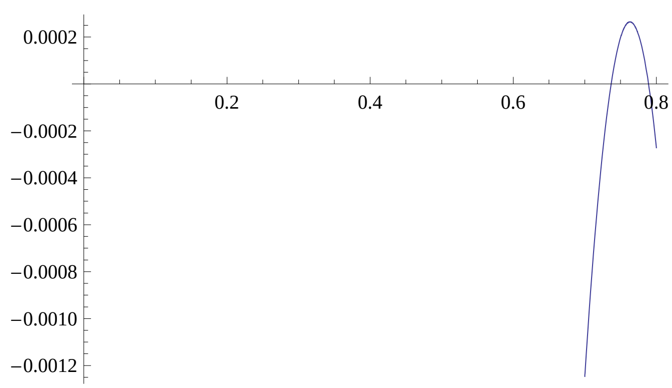

Figure 2: The plot of $F(0.4705, \cdot)$.

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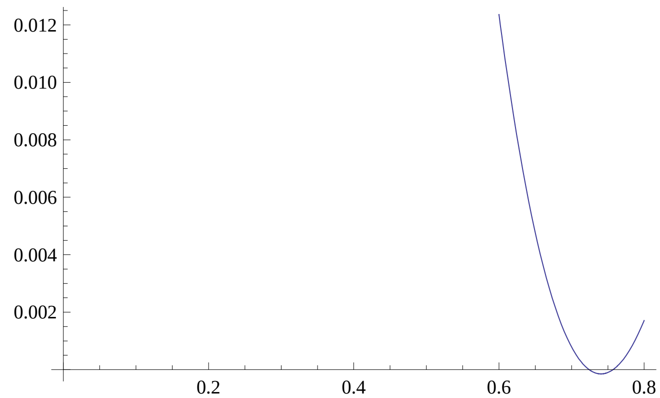

Figure 3: The plot of $G(0.59165, \cdot)$

References

[1] V. Bevc, J. L. Palmer and C. Süsskind, On the design of the transition region of axisymmetric, magnetically focused beam valves, **J. British Inst. Radio Engineer.** 18 (1958), 696–708.

[2] D. Bonheure, C. Fabry and D. Smets, Periodic solutions of forced isochronous oscillators at resonance. **Discrete Contin. Dyn. Syst.** 8 (2002), 907–930.

[3] A. Boscaggin and M. Garrione, Resonance and rotation numbers for planar Hamiltonian systems: multiplicity results via the Poincaré-Birkhoff theorem, **Nonlinear Anal.** 74 (2011), 4166–4185.

[4] H. Broer and M. Levi, Geometrical aspects of stability theory for Hill's equations, **Arch. Rational Mech. Anal.** 131 (1995), 225–240.

[5] A. Cabada and J. A. Cid, On comparison principles for the periodic Hill's equation, **J. Lond. Math. Soc.** (2) 86 (2012), 272–290.

[6] M. del Pino, R. Manásevich and A. Montero, T-periodic solutions for some second order differential equations with singularities, **Proc. Roy. Soc. Edinburgh Sect. A** 120 (1992), 231–243.

[7] T. Ding, A boundary value problem for the periodic Brillouin focusing system, **Acta Sci. Natur. Univ. Pekinensis** 11 (1965), 31–38 (in Chinese).

[8] C. Fabry, Periodic solutions of the equation $x'' + f(t, x) = 0$, **Séminaire de Mathématique** 117 (1987), Louvain-la-Neuve.

[9] C. Fabry and P. Habets, Periodic solutions of second order differential equations with superlinear asymmetric nonlinearities, **Arch. Math. (Basel)** 60 (1993), 266–276.

[10] A. Fonda and A. Sfecci, A general method for the existence of periodic solutions of differential systems in the plane, **J. Differential Equations** 252 (2012), 1369–1391.

[11] A. Fonda and R. Toader, Radially symmetric systems with a singularity and asymptotically linear growth, **Nonlinear Anal.** 74 (2011), 2485–2496.

[12] W. Magnus and S. Winkler, Hill's equation, corrected reprint of 1966 edition, Dover, New York, 1979.

[13] J. Ren, Z. Cheng and S. Siegmund, Positive periodic solution for Brillouin electron beam focusing systems, **Discrete Cont. Dyn. Syst. Ser. B** 16 (2011), 385–392.

[14] P.J. Torres, Existence and uniqueness of elliptic periodic solutions of the Brillouin electron beam focusing system, **Math. Methods Appl. Sci.** 23 (2000), 1139–1143.

[15] P.J. Torres, Twist solutions of a Hill's equation with singular term, **Adv. Nonlinear Stud.** 2 (2002), 279–287.

[16] P.J. Torres, Existence of one-signed periodic solutions of some second-order differential equations via a Krasnoselskii fixed point theorem, **J. Differential Equations** 190 (2003), 643–662.

[17] Q. Yao, Periodic positive solution to a class of singular second-order ordinary differential equations, **Acta Appl. Math.** 110 (2010), 871–883.

[18] Y. Ye and X. Wang, Nonlinear differential equations arising in the theory of electron beam focusing, **Acta Math. Appl. Sinica** 1 (1978), 13–41.

[19] M. Zhang, Periodic solutions of Liénard equations with singular forces of repulsive type, **J. Math. Anal. Appl.** 203 (1996), 254–269.

[20] M. Zhang, A relationship between the periodic and the Dirichlet BVPs of singular differential equations, **Proc. Roy. Soc. Edinburgh Sect. A** 128 (1998), 1099–1114.

[21] M. Zhang, Periodic solutions of equations of Emarkov-Pinney type, **Adv. Nonlinear Stud.** 6 (2006), 57–67.