J. Differential Equations 255 (2013) 2340–2362

Contents lists available at SciVerse ScienceDirect

Journal of Differential Equations

www.elsevier.com/locate/jde

Elliptic problems involving the fractional Laplacian in $\mathbb{R}^N$

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A R T I C L E   I N F O

Article history:
Received 18 October 2012
Revised 11 June 2013
Available online 8 July 2013

MSC:
primary 35J60, 35R11
secondary 35S30, 35J20

Keywords:
Existence
Fractional Sobolev spaces
Variational methods

A B S T R A C T

We study the existence and multiplicity of solutions for elliptic equations in $\mathbb{R}^N$, driven by a non-local integro-differential operator, which main prototype is the fractional Laplacian. The model under consideration, denoted by $(\mathscr{P}_\lambda)$, depends on a real parameter λ and involves two superlinear nonlinearities, one of which could be critical or even supercritical. The main theorem of the paper establishes the existence of three critical values of λ which divide the real line in different intervals, where $(\mathscr{P}_\lambda)$ admits no solutions, at least one nontrivial non-negative entire solution and two nontrivial non-negative entire solutions.

1. Introduction

In this paper we prove the existence and multiplicity of solutions for non-local integro-differential equations in $\mathbb{R}^N$, whose prototype is given by

$$(-\Delta)^s u + a(x)u = \lambda w(x)|u|^{q-2}u - h(x)|u|^{r-2}u \quad \text{in } \mathbb{R}^N, \qquad (\mathscr{P}_\lambda)$$

where $\lambda \in \mathbb{R}$, $0 < s < 1$, $2s < N$ and $(-\Delta)^s$ is the fractional Laplacian operator. Up to normalization factors, $(-\Delta)^s u$ is defined pointwise for x in $\mathbb{R}^N$ by

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0022-0396/$ – see front matter
http://dx.doi.org/10.1016/j.jde.2013.06.016

$$(-\Delta)^s u(x) = -\frac{1}{2}\int_{\mathbb{R}^N} \frac{u(x+y)+u(x-y)-2u(x)}{|y|^{N+2s}}\,dy,$$

along any rapidly decaying function u of class $C^\infty(\mathbb{R}^N)$, see Lemma 3.5 of [18].

The nonlinear terms in $(\mathscr{P}_\lambda)$ *are related to the main elliptic part by* the request that

$$2 < q < \min\{r, 2^*\}, \tag{1.1}$$

where $2^* = 2N/(N-2s)$ is the critical Sobolev exponent for $H^s(\mathbb{R}^N)$. *The coefficient a is supposed to be in* $L^\infty_{\mathrm{loc}}(\mathbb{R}^N)$ *and to satisfy for a.a.* $x \in \mathbb{R}^N$

$$\nu(x) = \max\{a(x), \left(1+|x|\right)^{-2s}\}, \quad a(x) \geqslant \kappa\,\nu(x), \tag{1.2}$$

for some constant $\kappa \in (0,1]$. *The weight w verifies*

$$w \in L^{\wp}\left(\mathbb{R}^N\right) \cap L^\sigma_{\mathrm{loc}}\left(\mathbb{R}^N\right), \quad \text{with } \wp = 2^*/\left(2^*-q\right), \quad \sigma > \wp, \tag{1.3}$$

while *h is a positive weight of class* $L^1_{\mathrm{loc}}(\mathbb{R}^N)$. Finally, *h and w are related by the condition*

$$\int_{\mathbb{R}^N} \left[\frac{w(x)^r}{h(x)^q}\right]^{1/(r-q)} dx = H \in \mathbb{R}^+. \tag{1.4}$$

The main result of the paper is

Theorem 1.1. *Under the above assumptions there exist* λ^*, λ^{**} *and* $\bar{\lambda}$, *with* $0 < \lambda^* \leqslant \lambda^{**} \leqslant \bar{\lambda}$ *such that Eq.* $(\mathscr{P}_\lambda)$ *admits*

(i) *only the trivial solution if* $\lambda < \lambda^*$;
(ii) *a nontrivial non-negative entire solution if and only if* $\lambda \geqslant \lambda^{**}$;
(iii) *at least two nontrivial non-negative entire solutions if* $\lambda > \bar{\lambda}$.

The definition of *entire solution* for $(\mathscr{P}_\lambda)$, as well as the proof of Theorem 1.1(i), are given in Section 2, after the introduction of the main solution space X. Some preliminary results for existence are presented in Section 3 and in Appendix A. The proof of Theorem 1.1(ii) is discussed in Section 4, while Theorem 1.1(iii) is proved in Section 5.

For standing wave solutions of fractional Schrödinger equations in $\mathbb{R}^N$ we refer to [20,22,32,13,28], [19, Section 5] and to the references therein. Models governed by unbounded potentials V are investigated in [14] and in its recent extension [27]. All these papers, however, deal with problems which are not directly comparable to $(\mathscr{P}_\lambda)$. The present work is more related to the results on general quasilinear elliptic problems given in [4]. Indeed, in [4], as a corollary of the main theorems, we proved under (1.4) that there exists $\lambda^* > 0$ such that

$$\begin{gathered} -\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) + a(x)|u|^{p-2}u = \lambda w(x)|u|^{q-2}u - h(x)|u|^{r-2}u \quad \text{in } \mathbb{R}^N, \\ 1 < p < N, \qquad \max\{2,p\} < q < \min\{r, p^*\}, \qquad p^* = \frac{Np}{N-p}, \end{gathered} \tag{$\mathscr{E}_\lambda$}$$

admits at least a nontrivial non-negative entire solution if and only if $\lambda \geqslant \lambda^*$. Theorem 1.1(ii) extends Theorem A of [4] to non-local integro-differential equations. It would be interesting to understand if $\lambda^* = \lambda^{**}$ in Theorem 1.1. This possible gap does not rise in [4]. Indeed, if u is a solution of $(\mathscr{E}_\lambda)$ also $|u|$ is. The situation is more delicate for $(\mathscr{P}_\lambda)$, since the fractional Laplacian itself does not guarantee the same property. Hence, it remains an open problem to establish whether $\lambda^* = \lambda^{**}$ in the non-local setting.

The extension of Theorem A of [4] to $(\mathscr{P}_\lambda)$ is not trivial and requires to overcome several difficulties which arise in the new context. In particular, the proof of the main preliminary Theorem 4.2 needs a special care.

Furthermore, Theorem 1.1(iii) is a complete extension of Theorem B of [4] to the non-local equation $(\mathscr{P}_\lambda)$ and its proof is based on a new strategy.

For previous related results in the local setting and in bounded domains we refer to [3,2,15,23] for the semilinear case and to [16] for the quasilinear case. We also refer to [24] for the semilinear case in $\mathbb{R}^N$. Actually, for semilinear elliptic equations assumption (1.4) first appears in the existence Theorem 1.1 of [2] for Dirichlet problems in bounded domains Ω, see also [26] for quasilinear equations in $\mathbb{R}^N$. In the existence Theorem 1.2 of [2] *Alama* and *Tarantello* use the weaker assumption that $w(w/h)^{(q-2)/(r-q)}$ is in $L^{N/2}(\Omega)$. It is still an open problem to produce nontrivial solutions of $(\mathscr{P}_\lambda)$ when $w(w/h)^{(q-2)/(r-q)} \in L^{N/2s}(\mathbb{R}^N)$ replaces (1.4) and of $(\mathscr{E}_\lambda)$ when $w(w/h)^{(q-p)/(r-q)} \in L^{N/p}(\mathbb{R}^N)$.

In the last years a great attention has been devoted to the study of fractional and non-local problems. For example, some of the most recent contributions on the existence of positive solutions for critical fractional Laplacian elliptic Dirichlet problems in bounded domains are given in [5], where the effects of lower order perturbations are considered. Comparison and regularity results and a priori estimates on the solutions of special fractional Laplacian elliptic boundary value problems in bounded domains are presented in [17], via symmetrization techniques. For the existence, non-existence, multiplicity and bifurcation of solutions for square root Laplacian Dirichlet problems in bounded domains with sign-changing weights we refer to [33]. A mountain pass theorem and applications to Dirichlet problems in bounded domains involving non-local integro-differential operators of fractional Laplacian type are given in [29]. Existence of positive solutions of concave–convex Dirichlet fractional Laplacian problems in bounded domains is proved in [8].

However, the interest in non-local integro-differential problems goes beyond the mathematical curiosity. Indeed, they have impressive applications in different fields, as the thin obstacle problem, optimization, finance, phase transitions, stratified materials, anomalous diffusion, crystal dislocation, deblurring and denoising of images, and so on. For further details we refer to [10,11,13,14,18,22,27,30–32] and the references therein.

Theorem 1.1 continues to hold when $(-\Delta)^s u$ in $(\mathscr{P}_\lambda)$ is replaced by any non-local integro-differential operator $\mathscr{L}_K u$, defined pointwise by

$$\mathscr{L}_K u(x) = -\frac{1}{2}\int_{\mathbb{R}^N} \big[u(x+y) + u(x-y) - 2u(x)\big] K(y)\,dy,$$

along any rapidly decaying function u of class $C^\infty(\mathbb{R}^N)$, where the *positive weight* $K : \mathbb{R}^N \setminus \{0\} \to \mathbb{R}^+$ satisfies the main properties

(k_1) *$gK \in L^1(\mathbb{R}^N)$, where $g(x) = \min\{1, |x|^2\}$;*
(k_2) *there exists $\gamma > 0$ such that $K(x) \geqslant \gamma |x|^{-(N+2s)}$ for all $x \in \mathbb{R}^N \setminus \{0\}$;*
(k_3) *$K(x) = K(-x)$ for all $x \in \mathbb{R}^N \setminus \{0\}$.*

Few details of the main changes, in passing from $(-\Delta)^s u$ to $\mathscr{L}_K u$ in $(\mathscr{P}_\lambda)$, are given in Appendix B. Of course $\mathscr{L}_K u$ reduces to the fractional Laplace operator $(-\Delta)^s u$ when $K(x) = |x|^{-(N+2s)}$.

2. Preliminaries and non-existence

Let $D^s(\mathbb{R}^N)$ denote the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the Gagliardo norm

$$[u]_s = \left(\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy\right)^{1/2}.$$

The embedding $D^s(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$ is continuous, that is

$$\|u\|_{2^*} \leqslant C_{2^*}[u]_s \quad \text{for all } u \in D^s\big(\mathbb{R}^N\big), \tag{2.1}$$

where $C_{2^*}^2 = c(N)\frac{s(1-s)}{(N-2s)}$ by Theorem 1 of [25], see also Theorem 1 of [7]. The space E denotes the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\|_E = \left([u]_s^2 + \int_{\mathbb{R}^N} \nu(x)|u|^2\,dx\right)^{1/2}.$$

Clearly, $\|\cdot\|_E$ is a Hilbertian norm induced by the inner product

$$\begin{aligned}\langle u, v\rangle_E &= \iint_{\mathbb{R}^{2N}} \frac{[u(x)-u(y)]\cdot[v(x)-v(y)]}{|x-y|^{N+2s}}\,dx\,dy + \int_{\mathbb{R}^N} \nu(x)u(x)v(x)\,dx \\ &= \langle u, v\rangle_s + \langle u, v\rangle_\nu.\end{aligned}$$

Finally, X is the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\| = \big(\|u\|_E^2 + \|u\|_{r,h}^2\big)^{1/2}, \quad \text{where } \|u\|_{r,h}^r = \int_{\mathbb{R}^N} h(x)|u|^r dx.$$

From now on B_R will denote the ball in $\mathbb{R}^N$ of center zero and radius $R > 0$.

Lemma 2.1. *The embeddings $X \hookrightarrow E \hookrightarrow D^s(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$ are continuous, with $[u]_s \leqslant \|u\|_E$ for all $u \in E$ and $\|u\|_E \leqslant \|u\|$ for all $u \in X$.*

Moreover, for any $R > 0$ and $p \in [1, 2^*)$ the embeddings $E \hookrightarrow\hookrightarrow L^p(B_R)$ and $X \hookrightarrow\hookrightarrow L^p(B_R)$ are compact.

Proof. The first two embeddings of the chain $X \hookrightarrow E \hookrightarrow D^s(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$ are obviously continuous, with $[u]_s \leqslant \|u\|_E$ for all $u \in E$ and $\|u\|_E \leqslant \|u\|$ for all $u \in X$. The continuity of the third embedding follows from (2.1), as recalled above.

Fix $R > 0$. By the first part of the lemma the embedding $E \hookrightarrow H^s(B_R)$ is continuous, since $0 < k_1 \leqslant \nu(x) \leqslant k_2$ for a.a. $x \in B_R$ and for some positive numbers k_1 and k_2 depending only on R, being $a \in L^\infty_{\text{loc}}(\mathbb{R}^N)$ by (1.2). The embedding $H^s(B_R) \hookrightarrow\hookrightarrow L^p(B_R)$ is compact for all $p \in [1, 2^*)$ by Corollary 7.2 of [18], and so the embeddings $E \hookrightarrow\hookrightarrow L^p(B_R)$ and $X \hookrightarrow\hookrightarrow L^p(B_R)$ are compact. □

From the structural assumptions (1.2)–(1.4) all the coefficients a, w, h in $(\mathscr{P}_\lambda)$ are weights in $\mathbb{R}^N$. We indicate with $L^2(\mathbb{R}^N, a) = (L^2(\mathbb{R}^N, a), \|\cdot\|_{2,a})$, $L^q(\mathbb{R}^N, w) = (L^q(\mathbb{R}^N, w), \|\cdot\|_{q,w})$ and $L^r(\mathbb{R}^N, h) = (L^r(\mathbb{R}^N, h), \|\cdot\|_{r,h})$ the corresponding weighted Lebesgue spaces, which are uniformly convex Banach spaces by Proposition A.6 of [4].

Lemma 2.2. *The embedding $D^s(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N, w)$ is continuous, with*

$$\|u\|_{q,w} \leqslant \mathfrak{C}_w[u]_s \quad \text{for all } u \in D^s\big(\mathbb{R}^N\big), \tag{2.2}$$

and $\mathfrak{C}_w = C_{2^*}\|w\|_\wp^{1/q} > 0$. The embeddings

$$E \hookrightarrow\hookrightarrow L^q(\mathbb{R}^N, w) \quad \textit{and} \quad X \hookrightarrow\hookrightarrow L^q(\mathbb{R}^N, w)$$

are compact. Furthermore, for all $u \in E$

$$[u]_s^2 + \|u\|_{2,a}^2 \geqslant \kappa \|u\|_E^2, \tag{2.3}$$

where κ is given in (1.2).

Proof. By (1.3), (2.1) and Hölder's inequality, for all $u \in D^s(\mathbb{R}^N)$,

$$\|u\|_{q,w} \leqslant \left(\int_{\mathbb{R}^N} w(x)^{\wp}\, dx \right)^{1/\wp q} \cdot \left(\int_{\mathbb{R}^N} |u|^{2^*}\, dx \right)^{1/2^*} \leqslant C_{2^*} \|w\|_{\wp}^{1/q} [u]_s,$$

that is, (2.2) holds.

In order to prove the last part of the lemma it is enough to show that $E \hookrightarrow\hookrightarrow L^q(\mathbb{R}^N, w)$, that is, that $\|u_n - u\|_{q,w} \to 0$ as $n \to \infty$ whenever $u_n \rightharpoonup u$ in E. By Hölder's inequality,

$$\int_{\mathbb{R}^N \setminus B_R} w(x)|u_n - u|^q\, dx \leqslant M \left(\int_{\mathbb{R}^N \setminus B_R} w(x)^{\wp}\, dx \right)^{1/\wp} = o(1)$$

as $R \to \infty$, being $w \in L^{\wp}(\mathbb{R}^N)$ by (1.3) and $M = \sup_n \|u_n - u\|_{2^*}^q < \infty$. For all $\varepsilon > 0$ there exists $R_\varepsilon > 0$ so large that $\sup_n \int_{\mathbb{R}^N \setminus B_{R_\varepsilon}} w(x)|u_n - u|^q\, dx < \varepsilon/2$. Moreover, by (1.3), Hölder's inequality and Lemma 2.1 as $n \to \infty$

$$\int_{B_{R_\varepsilon}} w(x)|u_n - u|^q\, dx \leqslant \|w\|_{L^\sigma(B_{R_\varepsilon})} \|u_n - u\|_{L^{\sigma' q}(B_{R_\varepsilon})}^q = o(1),$$

since $\sigma' q < 2^*$. Hence, there exists $N_\varepsilon > 0$ such that $\int_{B_{R_\varepsilon}} w(x)|u_n - u|^q\, dx < \varepsilon/2$ for all $n \geqslant N_\varepsilon$. In conclusion, for all $n \geqslant N_\varepsilon$

$$\|u_n - u\|_{q,w}^q = \int_{\mathbb{R}^N \setminus B_{R_\varepsilon}} w(x)|u_n - u|^q\, dx + \int_{B_{R_\varepsilon}} w(x)|u_n - u|^q\, dx < \varepsilon,$$

as required. Now, from (1.2) we directly get (2.3), being $\kappa \in (0, 1]$. □

We say that $u \in X$ is a (*weak*) *entire solution* of $(\mathscr{P}_\lambda)$ if

$$\langle u, \varphi \rangle_s + \int_{\mathbb{R}^N} a(x) u \varphi\, dx = \lambda \int_{\mathbb{R}^N} w(x)|u|^{q-2} u \varphi\, dx - \int_{\mathbb{R}^N} h(x)|u|^{r-2} u \varphi\, dx \tag{2.4}$$

for all $\varphi \in X$.

Hence the entire solutions of $(\mathscr{P}_\lambda)$ correspond to the critical points of the C^1 energy functional $\Phi_\lambda : X \to \mathbb{R}$, defined by

$$\Phi_\lambda(u) = \frac{1}{2}[u]_s^2 + \frac{1}{2}\|u\|_{2,a}^2 - \frac{\lambda}{q}\|u\|_{q,w}^q + \frac{1}{r}\|u\|_{r,h}^r,$$

see the next Lemma 3.4. Similarly, non-negative entire solutions of $(\mathscr{P}_\lambda)$ are the critical points of the C^1 functional

$$\Psi_\lambda(u) = \frac{1}{2}[u]_s^2 + \frac{1}{2}\|u\|_{2,a}^2 - \frac{\lambda}{q}\|u^+\|_{q,w}^q + \frac{1}{r}\|u\|_{r,h}^r,$$

well-defined for all $u \in X$, see the next Lemma 3.4. Indeed, both u^+ and $u^- \in X$ for all $u \in X$, being $|u^+(x) - u^+(y)| \leqslant |u(x) - u(y)|$ and $|u^-(x) - u^-(y)| \leqslant |u(x) - u(y)|$ for all $x, y \in \mathbb{R}^N$. Furthermore, for all $u \in X$

$$\langle u, u^- \rangle_s = \iint_{\mathbb{R}^{2N}} \frac{u^-(x)^2 + u^-(y)^2 - 2u^-(x)u(y)}{|x-y|^{N+2s}}\,dx\,dy,$$

since $\iint_{\mathbb{R}^{2N}} \frac{u(x)u^-(y)}{|x-y|^{N+2s}}\,dx\,dy = \iint_{\mathbb{R}^{2N}} \frac{u^-(x)u(y)}{|x-y|^{N+2s}}\,dx\,dy$. Therefore, if $u \in X$ is a critical point of Ψ_λ, then by (1.2)

$$\begin{aligned}
0 &= \langle u, u^- \rangle_s + \int_{\mathbb{R}^N} a(x)|u^-|^2\,dx + \int_{\mathbb{R}^N} h(x)|u^-|^r\,dx \\
&\geqslant \iint_{\mathbb{R}^{2N}} \frac{[u^-(x) - u^-(y)]^2 + 2u^-(x)u^+(y)}{|x-y|^{N+2s}}\,dx\,dy + \kappa\|u^-\|_{2,\nu}^2 \\
&\geqslant \kappa\|u^-\|_E^2 + 2\iint_{\mathbb{R}^{2N}} \frac{u^-(x)u^+(y)}{|x-y|^{N+2s}}\,dx\,dy \geqslant 0,
\end{aligned}$$

in other words $u^- = 0$ in E, that is, the critical point u of Ψ_λ is non-negative in $\mathbb{R}^N$.

Lemma 2.3. *If $u \in X \setminus \{0\}$ and $\lambda \in \mathbb{R}$ satisfy*

$$[u]_s^2 + \|u\|_{2,a}^2 + \|u\|_{r,h}^r = \lambda\|u\|_{q,w}^q, \tag{2.5}$$

then $\lambda > 0$ and

$$c_1\lambda^{1/(2-q)} \leqslant \|u\|_{q,w} \leqslant c_2\lambda^{r/2(r-q)}, \tag{2.6}$$

where

$$c_1 = \left(\kappa/\mathfrak{C}_w^2\right)^{1/(q-2)} \quad \text{and} \quad c_2 = \left[(r-q)\mathfrak{C}_w^2 H/r\kappa\right]^{1/2}.$$

Proof. Let $u \in X \setminus \{0\}$ and $\lambda \in \mathbb{R}$ satisfy (2.5). Then $0 < \kappa\|u\|_E^2 \leqslant \lambda\|u\|_{q,w}^q$ by (2.3), so that $\lambda > 0$ and moreover

$$\|u\|_{q,w}^2 \leqslant \mathfrak{C}_w^2\|u\|_E^2 \leqslant \frac{\lambda\mathfrak{C}_w^2}{\kappa}\|u\|_{q,w}^q \tag{2.7}$$

by (2.2). Using Young's inequality

$$t\tau \leqslant \frac{t^\alpha}{\alpha} + \frac{\tau^\beta}{\beta},$$

with $t = h(x)^{q/r}|u|^q \geqslant 0$, $\tau = \lambda w(x)h(x)^{-q/r} \geqslant 0$, $\alpha = r/q > 1$ and $\beta = r/(r-q) > 1$, we find

$$\lambda w(x)|u|^q \leqslant \frac{q}{r}h(x)|u|^r + \frac{r-q}{r}\left(\frac{\lambda w(x)}{h(x)^{q/r}}\right)^{r/(r-q)}.$$

Integration over $\mathbb{R}^N$ gives

$$\lambda\|u\|_{q,w}^q \leqslant \frac{q}{r}\|u\|_{r,h}^r + \frac{r-q}{r}H\lambda^{r/(r-q)}. \tag{2.8}$$

Thus, by (2.5) we obtain

$$[u]_s^2 + \|u\|_{2,a}^2 \leqslant \frac{q-r}{r}\|u\|_{r,h}^r + \frac{r-q}{r}H\lambda^{r/(r-q)} \leqslant \frac{r-q}{r}H\lambda^{r/(r-q)},$$

being $q < r$. Hence, since $u \not\equiv 0$ by assumption, the last inequality and (2.7) give (2.6), with c_1 and c_2 as stated. □

If $(\mathscr{P}_\lambda)$ admits a *nontrivial* entire solution $u \in X$, then $\lambda \geqslant \lambda_0$ by (2.6), where $\lambda_0 = (c_1/c_2)^{2(r-q)(q-2)/q(r-2)} > 0$. Define

$$\lambda^* = \sup\{\lambda > 0\colon\ (\mathscr{P}_\mu) \textit{ admits only the trivial solution for all } \mu < \lambda\}.$$

Theorem 1.1(i) follows directly by the definition of λ^*. Similarly, put

$$\lambda^*_{\Psi_\lambda} = \sup\{\lambda > 0\colon\ (\mathscr{P}_\mu) \textit{ admits no nontrivial non-negative solution for all } \mu < \lambda\}.$$

Clearly $\lambda^*_{\Psi_\lambda} \geqslant \lambda^* \geqslant \lambda_0 > 0$.

3. Preliminary results for existence

By the results of Section 2 from now on we consider only the case $\lambda > 0$.

Lemma 3.1. *The functionals Φ_λ and Ψ_λ are coercive in X. In particular, any sequence $(u_n)_n$ in X such that either $(\Phi_\lambda(u_n))_n$ or $(\Psi_\lambda(u_n))_n$ is bounded admits a weakly convergent subsequence in X.*

Proof. Let us consider the following elementary inequality: for every $k_1, k_2 > 0$ and $0 < \alpha < \beta$

$$k_1|t|^\alpha - k_2|t|^\beta \leqslant C_{\alpha\beta}k_1\left(\frac{k_1}{k_2}\right)^{\alpha/(\beta-\alpha)} \quad \text{for all } t \in \mathbb{R}, \tag{3.1}$$

where $C_{\alpha\beta} > 0$ is a constant depending only on α and β.

Taking $k_1 = \lambda w(x)/q$, $k_2 = h(x)/2r$, $\alpha = q$, $\beta = r$ and $t = u(x)$ in (3.1), for all $x \in \mathbb{R}^N$ we have

$$\frac{\lambda}{q}w(x)\left|u(x)\right|^q - \frac{h(x)}{2r}\left|u(x)\right|^r \leqslant \mathcal{C}\lambda^{r/(r-q)}\left[\frac{w(x)^r}{h(x)^q}\right]^{1/(r-q)},$$

where $\mathcal{C} = C_{qr}[2r/q]^{q/(r-q)}/q$. Integrating the above inequality over $\mathbb{R}^N$, we get by (1.4)

$$\frac{\lambda}{q}\|u\|_{q,w}^q - \frac{1}{2r}\|u\|_{r,h}^r \leqslant C_\lambda,$$

where $C_\lambda = \mathcal{C}H\lambda^{r/(r-q)} > 0$.

Therefore, by (2.3) for all $u \in X$

$$\begin{aligned}
\Phi_\lambda(u) &= \frac{1}{2}[u]_s^2 + \frac{1}{2}\|u\|_{2,a}^2 - \left[\frac{\lambda}{q}\|u\|_{q,w}^q - \frac{1}{2r}\|u\|_{r,h}^r\right] - \frac{1}{2r}\|u\|_{r,h}^r + \frac{1}{r}\|u\|_{r,h}^r \\
&\geqslant \frac{\kappa}{2}\|u\|_E^2 + \frac{1}{2r}\|u\|_{r,h}^r - C_\lambda \geqslant \frac{\kappa}{2}\|u\|_E^2 + \frac{1}{2r}\left(\|u\|_{r,h}^2 - 1\right) - C_\lambda \\
&\geqslant \frac{\min\{\kappa, r^{-1}\}}{2}\|u\|^2 - C_\lambda - \frac{1}{2r}.
\end{aligned}$$

Hence, Φ_λ is coercive in X. This implies at once that also Ψ_λ is coercive in X, being $\Psi_\lambda(u) \geqslant \Phi_\lambda(u)$ for all $u \in X$.

The last part of the claim follows at once by the coercivity of Φ_λ and Ψ_λ and the reflexivity of the space X, see Proposition A.1. □

Lemma 3.2. *The functional $\Psi : X \to \mathbb{R}$, $\Psi(u) = \frac{1}{2}[u]_s^2$, is convex and of class C^1. In particular, Ψ is weakly lower semicontinuous in X.*

Proof. The convexity is trivial. Now, let $(u_n)_n$, $u \in X$ be such that $u_n \to u$ in X. Then clearly $[u_n - u]_s \to 0$ as $n \to \infty$. Consider the following elementary inequality $||t|^2 - |\tau|^2| \leqslant 2(|t-\tau|^2 + |\tau| \cdot |t-\tau|)$ which is valid for all $t, \tau \in \mathbb{R}$. Applying this relation and Hölder's inequality, we have

$$\begin{aligned}
\left|\Psi(u_n) - \Psi(u)\right| &\leqslant \frac{1}{2}\iint_{\mathbb{R}^{2N}} \frac{||u_n(x) - u_n(y)|^2 - |u(x) - u(y)|^2|}{|x-y|^{N+2s}}\,dx\,dy \\
&\leqslant \iint_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y) - u(x) + u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy \\
&\quad + \iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)| \cdot |u_n(x) - u_n(y) - u(x) + u(y)|}{|x-y|^{N+2s}}\,dx\,dy \\
&\leqslant [u_n - u]_s^2 + [u]_s[u_n - u]_s = o(1)
\end{aligned}$$

as $n \to \infty$. This shows the continuity of Ψ.

Moreover, Ψ is Gâteaux-differentiable in X and for all $u, \varphi \in X$

$$\left\langle \Psi'(u), \varphi \right\rangle = \iint_{\mathbb{R}^{2N}} \frac{[u(x) - u(y)] \cdot [\varphi(x) - \varphi(y)]}{|x-y|^{N+2s}}\,dx\,dy = \langle u, \varphi \rangle_s.$$

Now, let $(u_n)_n$, $u \in X$ be such that $u_n \to u$ in X as $n \to \infty$. Of course

$$\left\|\Psi'(u_n) - \Psi'(u)\right\|_{X'} = \sup_{\substack{\varphi \in X \\ \|\varphi\|=1}} \left|\langle u_n - u, \varphi \rangle_s\right| \leqslant [u_n - u]_s \leqslant \|u_n - u\|,$$

that is, Ψ is of class C^1, as claimed. Finally, Ψ is weakly lower semicontinuous in X by Corollary 3.9 of [9]. □

For any $(x,u) \in \mathbb{R}^N \times \mathbb{R}$ put

$$f(x,u) = \lambda w(x)|u|^{q-2}u - h(x)|u|^{r-2}u, \tag{3.2}$$

so that

$$F(x,u) = \int_0^u f(x,v)\,dv = \frac{\lambda}{q} w(x)|u|^q - h(x)\frac{|u|^r}{r}. \tag{3.3}$$

Lemma 3.3. *For any fixed $u \in X$ the functional $\mathcal{F}_u : X \to \mathbb{R}$, defined by*

$$\mathcal{F}_u(v) = \int_{\mathbb{R}^N} f\big(x, u(x)\big) v(x)\,dx,$$

is in X'. In particular, if $v_n \rightharpoonup v$ in X then $\mathcal{F}_u(v_n) \to \mathcal{F}_u(v)$.

Proof. Take $u \in X$. Clearly $\mathcal{F}_u$ is linear. Moreover, using (2.2), we get for all $v \in X$

$$\begin{aligned}\big|\mathcal{F}_u(v)\big| &\leqslant \lambda \int_{\mathbb{R}^N} w(x)|u|^{q-1}|v|\,dx + \int_{\mathbb{R}^N} h(x)|u|^{r-1}|v|\,dx \\ &\leqslant \lambda\|u\|_{q,w}^{q-1}\|v\|_{q,w} + \|u\|_{r,h}^{r-1}\|v\|_{r,h} \leqslant \big(\lambda \mathfrak{C}_w\|u\|_{q,w}^{q-1} + \|u\|_{r,h}^{r-1}\big)\|v\|,\end{aligned}$$

and so $\mathcal{F}_u$ is continuous in X. □

In the next result we strongly use the assumption $q > 2$. An interesting open question occurs when $1 < q < 2 < r$, cf. Theorem 2.1 of [3] for homogeneous Dirichlet problems in bounded domains of $\mathbb{R}^N$.

Lemma 3.4. *The functionals Φ_λ and Ψ_λ are of class $C^1(X)$ and Φ_λ is sequentially weakly lower semicontinuous in X, that is, if $u_n \rightharpoonup u$ in X, then*

$$\Phi_\lambda(u) \leqslant \liminf_{n\to\infty} \Phi_\lambda(u_n). \tag{3.4}$$

Proof. Lemmas 3.2 and A.3–A.5 imply that Φ_λ and Ψ_λ are of class $C^1(X)$. Let $(u_n)_n$, $u \in X$ be such that $u_n \rightharpoonup u$ in X. The definition of Φ_λ and (3.2) give

$$\begin{aligned}\Phi_\lambda(u) - \Phi_\lambda(u_n) &= \frac{1}{2}\big([u]_s^2 - [u_n]_s^2 + \|u\|_{2,a}^2 - \|u_n\|_{2,a}^2\big) \\ &\quad + \int_{\mathbb{R}^N} \big[F(x,u_n) - F(x,u)\big]\,dx.\end{aligned} \tag{3.5}$$

Since $u_n \rightharpoonup u$ in X, Lemmas 3.2 and A.3 imply that

$$[u]_s^2 \leqslant \liminf_{n\to\infty}[u_n]_s^2 \quad \text{and} \quad \|u\|_{2,a}^2 \leqslant \liminf_{n\to\infty}\|u_n\|_{2,a}^2.$$

Hence, by (3.5)

$$\limsup_{n\to\infty}\big[\Phi_\lambda(u)-\Phi_\lambda(u_n)\big]\leqslant\limsup_{n\to\infty}\int_{\mathbb{R}^N}\big[F(x,u_n)-F(x,u)\big]\,dx. \tag{3.6}$$

By (3.2) and (3.3), for all $t\in[0,1]$,

$$\begin{aligned}F_u\big(x,u+t(u_n-u)\big)&=f\big(x,u+t(u_n-u)\big)\\&=f(x,u)+(u_n-u)\int_0^t f_u\big(x,u+\tau(u_n-u)\big)\,d\tau,\end{aligned} \tag{3.7}$$

where clearly

$$f_u(x,z)=\lambda(q-1)w(x)|z|^{q-2}-h(x)(r-1)|z|^{r-2}.$$

Multiplying (3.7) by u_n-u and integrating over $[0,1]$, we obtain

$$\begin{aligned}F(x,u_n)-F(x,u)&=f(x,u)(u_n-u)\\&\quad+(u_n-u)^2\int_0^1\left(\int_0^t f_u\big(x,u+\tau(u_n-u)\big)\,d\tau\right)dt.\end{aligned} \tag{3.8}$$

By (3.1), with $t=z$, $k_1=\lambda w(x)(q-1)$, $k_2=h(x)(r-1)$, $\alpha=q-2>0$ and $\beta=r-2>0$, we get

$$f_u(x,z)\leqslant 2C_1 w(x)^{2/q}\left[\frac{w(x)^{r/q}}{h(x)}\right]^{(q-2)/(r-q)},$$

where C_1 is a positive constant, depending only on q, r and λ. Consequently, (3.8) yields

$$\begin{aligned}\int_{\mathbb{R}^N}\big[F(x,u_n)-F(x,u)\big]\,dx&\leqslant\int_{\mathbb{R}^N}f(x,u)(u_n-u)\,dx\\&\quad+C_1\int_{\mathbb{R}^N}w(x)^{2/q}(u_n-u)^2\left[\frac{w(x)^{r/q}}{h(x)}\right]^{(q-2)/(r-q)}dx\\&\leqslant\int_{\mathbb{R}^N}f(x,u)(u_n-u)\,dx+C_1H^{(q-2)/q}\|u_n-u\|_{q,w}^2,\end{aligned} \tag{3.9}$$

by Hölder's inequality and (1.4). Now, Lemma 3.3 gives

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}f(x,u)(u_n-u)\,dx=0, \tag{3.10}$$

and Lemma 2.2 implies

$$\lim_{n\to\infty} \|u_n - u\|_{q,w} = 0. \tag{3.11}$$

Combining (3.9)–(3.11) with (3.6) we get the claim (3.4). □

4. Existence if λ is large

Define

$$\bar{\lambda} = \inf_{\substack{u\in X \\ \|u\|_{q,w}=1}} \left\{ \frac{q}{2}[u]_s^2 + \frac{q}{2}\|u\|_{2,a}^2 + \frac{q}{r}\|u\|_{r,h}^r \right\}.$$

Note that $\bar{\lambda} > 0$. Indeed, for any $u \in X$ with $\|u\|_{q,w} = 1$, by Hölder's inequality and (1.4), we have

$$1 = \|u\|_{q,w}^q = \int_{\mathbb{R}^N} \frac{w(x)}{h(x)^{q/r}} h(x)^{q/r} |u|^q \, dx \leqslant H^{(r-q)/r} \|u\|_{r,h}^q.$$

Consequently, using also (2.3) and Lemma 2.1, we get

$$\frac{q}{2}[u]_s^2 + \frac{q}{2}\|u\|_{2,a}^2 + \frac{q}{r}\|u\|_{r,h}^r \geqslant \frac{\kappa q}{2}\|u\|_E^2 + \frac{q}{r} H^{(q-r)/q} \geqslant \frac{\kappa q}{2\mathfrak{C}_w^2} + \frac{q}{r} H^{(q-r)/q},$$

where $\mathfrak{C}_w > 0$ is given in (2.2). In other words,

$$\bar{\lambda} \geqslant \frac{\kappa q}{2\mathfrak{C}_w^2} + \frac{q}{r} H^{(q-r)/q} > 0.$$

Lemma 4.1. *For all $\lambda > \bar{\lambda}$ there exists a global nontrivial non-negative minimizer $e \in X$ of Φ_λ with negative energy, that is $\Phi_\lambda(e) < 0$. Furthermore, e is also a critical point of Ψ_λ and $\Psi_\lambda(e) = \Phi_\lambda(e) < 0$.*

Proof. For all $\lambda > 0$ the functional Φ_λ is sequentially weakly lower semicontinuous, bounded below and coercive in the reflexive Banach space X by Lemmas 3.1, 3.4 and Proposition A.1. Hence, Theorem 6.1.1 of [6] implies that for all $\lambda > 0$ there exists a global minimizer $e \in X$ of Φ_λ, that is

$$\Phi_\lambda(e) = \inf_{v\in X} \Phi_\lambda(v).$$

Clearly e is a solution of $(\mathscr{P}_\lambda)$. We prove that $e \not\equiv 0$ whenever $\lambda > \bar{\lambda}$, showing that $\inf_{v\in X} \Phi_\lambda(v) < 0$.

Let $\lambda > \bar{\lambda}$. Then there exists a function $\varphi \in X$, with $\|\varphi\|_{q,w} = 1$, such that

$$\lambda \|\varphi\|_{q,w}^q = \lambda > \frac{q}{2}\left([\varphi]_s^2 + \|\varphi\|_{2,a}^2\right) + \frac{q}{r}\|\varphi\|_{r,h}^r.$$

This can be rewritten as

$$\Phi_\lambda(\varphi) = \frac{1}{2}\left([\varphi]_s^2 + \|\varphi\|_{2,a}^2\right) - \frac{\lambda}{q}\|\varphi\|_{q,w}^q + \frac{1}{r}\|\varphi\|_{r,h}^r < 0$$

and consequently $\Phi_\lambda(e) = \inf_{v\in X} \Phi_\lambda(v) \leqslant \Phi_\lambda(\varphi) < 0$.

Hence, for any $\lambda > \bar{\lambda}$ Eq. $(\mathscr{P}_\lambda)$ has a nontrivial entire solution $e \in X$ such that $\Phi_\lambda(e) < 0$. Finally, we may assume $e \geqslant 0$ in $\mathbb{R}^N$. Indeed, $|e| \in X$ and $\Phi_\lambda(|e|) \leqslant \Phi_\lambda(e)$, being $[|u|]_s \leqslant [u]_s$. This gives $\Phi_\lambda(e) = \Phi_\lambda(|e|)$, due to the minimality of e.

The second part of the lemma is almost trivial, being $\Phi_\lambda(u) = \Psi_\lambda(u)$ for all $u \in X$, with $u \geqslant 0$ in $\mathbb{R}^N$, so that e is also a nontrivial global minimizer of Ψ_λ in X. □

Define

$$\lambda^{**} = \inf\big\{\lambda > 0\colon\ (\mathscr{P}_\lambda)\ \textit{admits a nontrivial non-negative entire solution}\big\}.$$

Lemma 4.1 assures that this definition is meaningful and that $\bar{\lambda} \geqslant \lambda^{**}$.

Theorem 4.2. *For any $\lambda > \lambda^{**}$ Eq. $(\mathscr{P}_\lambda)$ admits a nontrivial non-negative entire solution $u_\lambda \in X$.*

Proof. Fix $\lambda > \lambda^{**}$. By definition of λ^{**} there exists $\mu \in (\lambda^{**}, \lambda)$ such that Φ_μ has a nontrivial critical point $u_\mu \in X$, with $u_\mu \geqslant 0$ in $\mathbb{R}^N$. Of course, u_μ is a subsolution for $(\mathscr{P}_\lambda)$. Consider the following minimization problem

$$\inf_{v \in \mathcal{M}} \Phi_\lambda(v), \qquad \mathcal{M} = \{v \in X\colon\ v \geqslant u_\mu\}.$$

First note that $\mathcal{M}$ is closed and convex, and in turn also weakly closed. Moreover, as shown in the proof of Lemma 4.1, Theorem 6.1.1 of [6] can be applied in X and so in the weakly closed set $\mathcal{M}$. Hence, Φ_λ attains its infimum in $\mathcal{M}$, i.e. there exists $u_\lambda \geqslant u_\mu$ such that $\Phi_\lambda(u_\lambda) = \inf_{v \in \mathcal{M}} \Phi_\lambda(v)$.

We claim that u_λ is a solution of $(\mathscr{P}_\lambda)$, which is clearly non-negative. Indeed, take $\varphi \in C_0^\infty(\mathbb{R}^N)$ and $\varepsilon > 0$. Put

$$\varphi_\varepsilon = \max\{0, u_\mu - u_\lambda - \varepsilon\varphi\} \geqslant 0 \quad \text{and} \quad v_\varepsilon = u_\lambda + \varepsilon\varphi + \varphi_\varepsilon,$$

so that $v_\varepsilon \in \mathcal{M}$. Of course

$$0 \leqslant \big\langle \Phi'_\lambda(u_\lambda), v_\varepsilon - u_\lambda\big\rangle = \varepsilon\big\langle \Phi'_\lambda(u_\lambda), \varphi\big\rangle + \big\langle \Phi'_\lambda(u_\lambda), \varphi_\varepsilon\big\rangle,$$

and in turn

$$\big\langle \Phi'_\lambda(u_\lambda), \varphi\big\rangle \geqslant -\frac{1}{\varepsilon}\big\langle \Phi'_\lambda(u_\lambda), \varphi_\varepsilon\big\rangle. \tag{4.1}$$

Since u_μ is a subsolution of $(\mathscr{P}_\lambda)$ and $\varphi_\varepsilon \geqslant 0$ we get that $\langle \Phi'_\lambda(u_\mu), \varphi_\varepsilon\rangle \leqslant 0$. In particular,

$$\big\langle \Phi'_\lambda(u_\lambda), \varphi_\varepsilon\big\rangle = \big\langle \Phi'_\lambda(u_\mu), \varphi_\varepsilon\big\rangle + \big\langle \Phi'_\lambda(u_\lambda) - \Phi'_\lambda(u_\mu), \varphi_\varepsilon\big\rangle \leqslant \big\langle \Phi'_\lambda(u_\lambda) - \Phi'_\lambda(u_\mu), \varphi_\varepsilon\big\rangle.$$

Define $\Omega_\varepsilon = \{x \in \mathbb{R}^N\colon\ u_\lambda(x) + \varepsilon\varphi(x) \leqslant u_\mu(x) < u_\lambda(x)\}$. Clearly Ω_ε is a subset of $\operatorname{supp}\varphi$. Put $u = u_\lambda - u_\mu$ and

$$\mathscr{U}_\varepsilon(x, y) = \frac{[u(x) - u(y)] \cdot [\varphi_\varepsilon(x) - \varphi_\varepsilon(y)]}{|x - y|^{N+2s}},$$

so that

$$\begin{aligned}
\langle u, \varphi_\varepsilon\rangle_s &= \iint\limits_{\Omega_\varepsilon\times\Omega_\varepsilon} \mathscr{U}_\varepsilon(x,y)\,dx\,dy + \iint\limits_{\Omega_\varepsilon\times(\mathbb{R}^N\setminus\Omega_\varepsilon)} \mathscr{U}_\varepsilon(x,y)\,dx\,dy + \iint\limits_{(\mathbb{R}^N\setminus\Omega_\varepsilon)\times\Omega_\varepsilon} \mathscr{U}_\varepsilon(x,y)\,dx\,dy \\
&= \iint\limits_{\Omega_\varepsilon\times\Omega_\varepsilon} \mathscr{U}_\varepsilon(x,y)\,dx\,dy + 2\iint\limits_{\Omega_\varepsilon\times(\mathbb{R}^N\setminus\Omega_\varepsilon)} \mathscr{U}_\varepsilon(x,y)\,dx\,dy \\
&\leqslant -\varepsilon\Bigg(\iint\limits_{\Omega_\varepsilon\times\Omega_\varepsilon} \mathscr{U}(x,y)\,dx\,dy + 2\iint\limits_{\Omega_\varepsilon\times(\mathbb{R}^N\setminus\Omega_\varepsilon)} \mathscr{U}(x,y)\,dx\,dy\Bigg) \\
&\leqslant 2\varepsilon \iint\limits_{\Omega_\varepsilon\times\mathbb{R}^N} \big|\mathscr{U}(x,y)\big|\,dx\,dy,
\end{aligned}$$

where similarly $\mathscr{U}(x,y) = \frac{[u(x)-u(y)]\cdot[\varphi(x)-\varphi(y)]}{|x-y|^{N+2s}}$. Using the notation of (3.2), we get

$$\Bigg|\int\limits_{\Omega_\varepsilon}\big(f(x,u_\lambda)-f(x,u_\mu)\big)\big(-u(x)-\varepsilon\varphi(x)\big)\,dx\Bigg| \leqslant \varepsilon\int\limits_{\Omega_\varepsilon}\big|f(x,u_\lambda)-f(x,u_\mu)\big|\cdot\big|\varphi(x)\big|\,dx,$$

since $0 \leqslant -u-\varepsilon\varphi = u_\mu - u_\lambda + \varepsilon|\varphi| < \varepsilon|\varphi|$ in Ω_ε. Therefore,

$$\begin{aligned}
\big\langle \Phi_\lambda'(u_\lambda), \varphi_\varepsilon\big\rangle \leqslant \varepsilon\Bigg(2\iint\limits_{\Omega_\varepsilon\times\mathbb{R}^N}\big|\mathscr{U}(x,y)\big|\,dx\,dy + \int\limits_{\Omega_\varepsilon} a(x)u(x)\big|\varphi(x)\big|\,dx \\
+ \int\limits_{\Omega_\varepsilon}\big|f(x,u_\lambda)-f(x,u_\mu)\big|\cdot\big|\varphi(x)\big|\,dx\Bigg).
\end{aligned}$$

Hence,

$$\big\langle \Phi_\lambda'(u_\lambda), \varphi_\varepsilon\big\rangle \leqslant \varepsilon\Bigg(\int\limits_{\Omega_\varepsilon}\psi(x)\,dx + 2\iint\limits_{\Omega_\varepsilon\times\mathbb{R}^N}\big|\mathscr{U}(x,y)\big|\,dx\,dy\Bigg), \tag{4.2}$$

where $\psi(x) = \{a(u_\lambda - u_\mu) + |f(x,u_\lambda) - f(x,u_\mu)|\}|\varphi|$. We claim that ψ is in $L^1(\operatorname{supp}\varphi)$. Indeed, au_λ, au_μ and also $|f(x,u_\lambda)-f(x,u_\mu)|$ are in $L^1_{\mathrm{loc}}(\mathbb{R}^N)$, being

$$\big|f(x,u_\lambda)-f(x,u_\mu)\big| \leqslant \lambda w(x)\big(u_\lambda^{q-1}+u_\mu^{q-1}\big) + h(x)\big(u_\lambda^{r-1}+u_\mu^{r-1}\big).$$

In fact, $a \in L^{N/2s}(\operatorname{supp}\varphi)$, since $a \in L^\infty_{\mathrm{loc}}(\mathbb{R}^N)$ by (1.2), so that by Hölder's inequality

$$\int\limits_{\operatorname{supp}\varphi} a(x)u_\lambda\,dx \leqslant |\operatorname{supp}\varphi|^{1/2^*}\Bigg(\int\limits_{\operatorname{supp}\varphi} a(x)^{N/2s}\,dx\Bigg)^{2s/N}\|u_\lambda\|_{2^*} = C_1 \tag{4.3}$$

and $C_1 = C_1(\operatorname{supp}\varphi)$. Similarly, by Hölder's inequality and (1.3), we obtain

$$\int\limits_{\operatorname{supp}\varphi} w(x)u_\lambda^{q-1}\,dx \leqslant |\operatorname{supp}\varphi|^{1/2^*}\|w\|_{\wp}\|u_\lambda\|_{2^*}^{q-1} = C_2, \tag{4.4}$$

and $C_2 = C_2(\operatorname{supp}\varphi)$. Finally, since $h \in L^1_{\rm loc}(\mathbb{R}^N)$ and $u_\lambda \in L^r(\mathbb{R}^N, h)$, then

$$\int\limits_{\operatorname{supp}\varphi} h(x)u_\lambda^{r-1}\,dx \leqslant \Bigl(\,\int\limits_{\operatorname{supp}\varphi} h(x)\,dx\Bigr)^{1/r} \|u_\lambda\|_{r,h}^{r-1} = C_3, \tag{4.5}$$

with $C_3 = C_3(\operatorname{supp}\varphi)$. The estimates (4.3)–(4.5) hold also for u_μ. The claim is so proved.

We next show that

$$\lim_{\varepsilon\to 0^+}\Bigl(\,\int\limits_{\Omega_\varepsilon}\psi(x)\,dx + 2\iint\limits_{\Omega_\varepsilon\times\mathbb{R}^N}\bigl|\mathscr{U}(x,y)\bigr|\,dx\,dy\Bigr) = 0. \tag{4.6}$$

Indeed, $\int_{\Omega_\varepsilon}\psi(x)\,dx = o(1)$, since $|\Omega_\varepsilon| \to 0$ as $\varepsilon \to 0^+$, $\Omega_\varepsilon \subset \operatorname{supp}\varphi$ and $\psi \in L^1(\operatorname{supp}\varphi)$. Similarly, $X \hookrightarrow D^s(\mathbb{R}^N)$ by Lemma 2.1, so that

$$\mathscr{U}(x,y) = \frac{[u_\lambda(x) - u_\mu(x) - u_\lambda(y) + u_\mu(y)]\cdot[\varphi(x)-\varphi(y)]}{|x-y|^{N+2s}} \in L^1\bigl(\mathbb{R}^{2N}\bigr).$$

Thus for all $\eta > 0$ there exists R_η so large that

$$\iint\limits_{(\operatorname{supp}\varphi)\times(\mathbb{R}^N\setminus B_{R_\eta})}\bigl|\mathscr{U}(x,y)\bigr|\,dx\,dy < \eta/2.$$

Since $|\Omega_\varepsilon \times B_{R_\eta}| \to 0$ as $\varepsilon \to 0^+$ and $\mathscr{U} \in L^1(\mathbb{R}^{2N})$ then there exist $\delta_\eta > 0$ and $\varepsilon_\eta > 0$ such that for all $\varepsilon \in (0, \varepsilon_\eta]$

$$|\Omega_\varepsilon \times B_{R_\eta}| < \delta_\eta \quad\text{and}\quad \iint\limits_{\Omega_\varepsilon\times B_{R_\eta}}\bigl|\mathscr{U}(x,y)\bigr|\,dx\,dy < \eta/2.$$

Therefore, for all $\varepsilon \in (0, \varepsilon_\eta]$

$$\iint\limits_{\Omega_\varepsilon\times\mathbb{R}^N}\bigl|\mathscr{U}(x,y)\bigr|\,dx\,dy < \eta,$$

being $\Omega_\varepsilon \subset \operatorname{supp}\varphi$. Hence (4.6) holds.

In conclusion, by (4.1), (4.2) and (4.6) it follows that $\langle\Phi_\lambda'(u_\lambda),\varphi\rangle \geqslant o(1)$ as $\varepsilon \to 0^+$. Therefore, $\langle\Phi_\lambda'(u_\lambda),\varphi\rangle \geqslant 0$ for all $\varphi \in C_0^\infty(\mathbb{R}^N)$, that is $\langle\Phi_\lambda'(u_\lambda),\varphi\rangle = 0$ for all $\varphi \in C_0^\infty(\mathbb{R}^N)$. Since $X = \overline{C_0^\infty(\mathbb{R}^N)}^{\|\cdot\|}$, we obtain that u_λ is a nontrivial non-negative solution of $(\mathscr{P}_\lambda)$. $\square$

Theorem 4.3. *$(\mathscr{P}_{\lambda^{**}})$ admits a nontrivial non-negative entire solution in X.*

Proof. Let $(\lambda_n)_n$ be a strictly decreasing sequence converging to λ^{**} and $u_n \in X$ be a nontrivial non-negative entire solution of $(\mathscr{P}_{\lambda_n})$. By (2.4) we get for all $\varphi \in X$

$$\langle u_n, \varphi\rangle_s = \int\limits_{\mathbb{R}^N} g_n\varphi\,dx, \tag{4.7}$$

where $n \mapsto g_n(x) = -a(x)u_n + \lambda_n w(x)|u_n|^{q-2}u_n - h(x)|u_n|^{r-2}u_n$. By (2.3)–(2.6) and the monotonicity of $(\lambda_n)_n$, we obtain

$$\kappa\|u_n\|_E^2 + \|u_n\|_{r,h}^r \leqslant \lambda_n\|u_n\|_{q,w}^q \leqslant c_2^q \lambda_1^{1+rq/2(r-q)}.$$

Therefore $(\|u_n\|_E)_n$ and $(\|u_n\|_{r,h})_n$ are bounded, and in turn also $(\|u_n\|)_n$ is bounded. By Lemma 2.2, Propositions A.1, A.2 and the fact that $L^q(\mathbb{R}^N, w)$ and $L^r(\mathbb{R}^N, h)$ are uniformly convex Banach spaces by Proposition A.6 of [4], it is possible to extract a subsequence, still relabeled $(u_n)_n$, satisfying

$$\begin{aligned} &u_n \rightharpoonup u \quad \text{in } X; && u_n \to u \quad \text{in } L^q\big(\mathbb{R}^N, w\big); \\ &u_n \rightharpoonup u \quad \text{in } L^r\big(\mathbb{R}^N, h\big); && u_n \to u \quad \text{a.e. in } \mathbb{R}^N, \end{aligned} \tag{4.8}$$

for some $u \in X$. Of course $u \geqslant 0$ a.e. in $\mathbb{R}^N$ and we claim that u is the solution we are looking for.

To this aim, first note that for all $\varphi \in X$

$$\langle u_n, \varphi\rangle_s \to \langle u, \varphi\rangle_s, \qquad \int_{\mathbb{R}^N} w(x)|u_n|^{q-2}u_n\varphi\, dx \to \int_{\mathbb{R}^N} w(x)|u|^{q-2}u\varphi\, dx, \tag{4.9}$$

as $n \to \infty$, since $u_n \rightharpoonup u$ in X and $u_n \to u$ in $L^q(\mathbb{R}^N, w)$. Furthermore, Lemmas A.3 and A.5 yield in particular the validity of (A.1) and (A.2) for all $\varphi \in X$. In conclusion, passing to the limit in (4.7) as $n \to \infty$, we get

$$\langle u, \varphi\rangle_s = -\int_{\mathbb{R}^N} a(x)u\varphi\, dx + \lambda^{**}\int_{\mathbb{R}^N} w(x)|u|^{q-2}u\varphi\, dx - \int_{\mathbb{R}^N} h(x)|u|^{r-2}u\varphi\, dx$$

for all $\varphi \in X$, that is, u is a non-negative entire solution of $(\mathscr{P}_{\lambda^{**}})$.

We finally claim that $u \not\equiv 0$. Indeed, $\|u\|_{q,w} = \lim_{n\to\infty}\|u_n\|_{q,w}$, since $u_n \to u$ in $L^q(\mathbb{R}^N, w)$ by (4.8). Moreover, (2.6) applied to each $u_n \neq 0$ implies that $\|u_n\|_{q,w} \geqslant c_1\lambda_n^{1/(2-q)}$, that is

$$\|u\|_{q,w} = \lim_{n\to\infty}\|u_n\|_{q,w} \geqslant c_1\big(\lambda^{**}\big)^{1/(2-q)} > 0,$$

since $\lambda_n \searrow \lambda^{**}$ and $\lambda^{**} > 0$. Hence $u \not\equiv 0$. $\square$

Theorems 4.2 and 4.3 guarantee that $\lambda^{**} = \lambda^*_{\Psi_\lambda}$. In particular, for all $\lambda \geqslant \lambda^{**}$ the nontrivial non-negative entire solution $u \in X$ constructed in Theorems 4.2 and 4.3 is a nontrivial critical point also of Ψ_λ.

Proof of Theorem 1.1(ii). The existence of λ^{**} follows from Lemma 4.1 and clearly $0 < \lambda^* \leqslant \lambda^{**}$. Now, if $(\mathscr{P}_\lambda)$ admits a nontrivial non-negative entire solution, then necessarily $\lambda \geqslant \lambda^{**}$ by definition of λ^{**}. On the other hand, Theorems 4.2 and 4.3 assure that $(\mathscr{P}_\lambda)$ admits a nontrivial non-negative entire solution for all $\lambda \geqslant \lambda^{**}$. $\square$

5. Existence of a second nontrivial non-negative entire solution

In this section we prove Theorem 1.1(iii). In particular, we show that if $\lambda > \bar{\lambda}$ Eq. $(\mathscr{P}_\lambda)$ admits the nontrivial non-negative global minimizer e, constructed in Lemma 4.1, and a second independent nontrivial non-negative entire solution $u \neq e$, via variational methods. We start by recalling a modification of the mountain pass theorem of *Ambrosetti* and *Rabinowitz*, established in [4], which involves two general Banach spaces X and E.

Theorem 5.1. *(See Theorem A.3 of [4].) Let $(X, \|\cdot\|)$ and $(E, \|\cdot\|_E)$ be two Banach spaces such that $X \hookrightarrow E$. Let $\Phi : X \to \mathbb{R}$ be a C^1 functional with $\Phi(0) = 0$. Suppose that there exist ϱ, $\alpha > 0$ and $e \in X$ such that $\|e\|_E > \varrho$, $\Phi(e) < \alpha$ and $\Phi(u) \geqslant \alpha$ for all $u \in X$ with $\|u\|_E = \varrho$.*

Then there exists a sequence $(u_n)_n$ in X such that for all n

$$c \leqslant \Phi(u_n) \leqslant c + \frac{1}{n^2} \quad \text{and} \quad \big\|\Phi'(u_n)\big\|_{X'} \leqslant \frac{2}{n},$$

where

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \Phi\big(\gamma(t)\big) \quad \text{and} \quad \Gamma = \big\{\gamma \in C\big([0,1]; X\big)\colon\ \gamma(0) = 0,\ \gamma(1) = e\big\}.$$

The proof of Theorem 5.1 is based on the *Ekeland variational principle*, see for instance [21]. For a similar generalization of the mountain pass theorem, obtained with a different proof and the use of the Palais–Smale compactness condition, we refer to Theorem 2.5 of [12].

We now show that for all $\lambda > 0$ the energy functional Ψ_λ satisfies the geometrical structure of Theorem 5.1.

Lemma 5.2. *For any $e \in X \setminus \{0\}$ and $\lambda > 0$ there exist $\varrho \in (0, \|e\|_E)$ and $\alpha = \alpha(\varrho) > 0$ such that $\Psi_\lambda(u) \geqslant \alpha$ for all $u \in X$, with $\|u\|_E = \varrho$.*

Proof. Let u be in X. By (2.2) and (2.3)

$$\Psi_\lambda(u) \geqslant \frac{\kappa}{2}\|u\|_E^2 - \frac{\lambda}{q}\big\|u^+\big\|_{q,w}^q \geqslant \frac{\kappa}{2}\|u\|_E^2 - \frac{\lambda}{q}\|u\|_{q,w}^q \geqslant \left(\frac{\kappa}{2} - \frac{\lambda}{q}\mathfrak{C}_w^q\|u\|_E^{q-2}\right)\|u\|_E^2.$$

Therefore, it is enough to take $0 < \varrho < \min\{(\kappa q/\lambda 2\mathfrak{C}_w^q)^{1/(q-2)}, \|e\|_E\}$, so that $\alpha = (\kappa/2 - \lambda\mathfrak{C}_w^q\varrho^{q-2}/q)\varrho^2 > 0$ satisfies the assertion. □

Proof of Theorem 1.1(iii). Lemma 4.1 shows that for all $\lambda > \bar{\lambda}$ there exists a nontrivial non-negative entire solution $e \in X$ of $(\mathscr{P}_\lambda)$, which is a global minimizer for Φ_λ in X. Hence e is also a global minimizer for Ψ_λ in X and $\Psi_\lambda(e) = \Phi_\lambda(e) < 0$.

Our aim now is to apply Theorem 5.1 to the functional Ψ_λ in order to find a second nontrivial non-negative entire solution of $(\mathscr{P}_\lambda)$, when $\lambda > \bar{\lambda}$.

We recall that Ψ_λ is of class C^1 by Lemmas 3.2 and A.3–A.5. Moreover, by Lemma 5.2 and Theorem 5.1 for all $\lambda > \bar{\lambda}$ there exists a sequence $(u_n)_n$ in X such that

$$\Psi_\lambda(u_n) \to c \quad \text{and} \quad \big\|\Psi_\lambda'(u_n)\big\|_{X'} \to 0$$

as $n \to \infty$, where

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \Psi_\lambda\big(\gamma(t)\big) \quad \text{and} \quad \Gamma = \big\{\gamma \in C\big([0,1]; X\big)\colon\ \gamma(0) = 0,\ \gamma(1) = e\big\}.$$

By Lemma 3.1 the sequence $(u_n)_n$ is bounded in X. From now on we can follow the argument of the proof of Theorem 4.3. We report here the main differences. By Lemmas 2.1, 2.2, Propositions A.1, A.2 and Propositions A.6, A.7 of [4] it is again possible to extract a subsequence, still relabeled $(u_n)_n$, satisfying (4.8). Moreover, from (4.8) it follows also that $u_n^+ \to u^+$ in $L^q(\mathbb{R}^N, w)$ being $\|u_n^+ - u^+\|_{q,w} \leqslant \|u_n - u\|_{q,w}$. We shall next prove that u is a nontrivial non-negative entire solution of $(\mathscr{P}_\lambda)$, with $u \neq e$.

Clearly, for any $\varphi \in X$

$$\big\langle \Psi_\lambda'(u_n), \varphi\big\rangle = \langle u_n, \varphi\rangle_s - \int\limits_{\mathbb{R}^N} g_n\varphi\, dx, \tag{5.1}$$

where here $n \mapsto g_n(x) = -a(x)u_n + \lambda w(x)|u_n^+|^{q-2}u_n^+ - h(x)|u_n|^{r-2}u_n$. Now,

$$\langle u_n, \varphi\rangle_s \to \langle u, \varphi\rangle_s, \qquad \int\limits_{\mathbb{R}^N} w(x)\big|u_n^+\big|^{q-2}u_n^+\varphi\, dx \to \int\limits_{\mathbb{R}^N} w(x)\big|u^+\big|^{q-2}u^+\varphi\, dx$$

as $n \to \infty$, since $u_n \rightharpoonup u$ in X and $u_n^+ \to u^+$ in $L^q(\mathbb{R}^N, w)$.

Moreover, Lemmas A.3 and A.5 give (A.1) and (A.2) for all $\varphi \in X$. Hence, passing to the limit as $n \to \infty$ in (5.1), we have

$$\langle u, \varphi\rangle_s + \int\limits_{\mathbb{R}^N} a(x)u\varphi\, dx = \lambda \int\limits_{\mathbb{R}^N} w(x)\big|u^+\big|^{q-2}u^+\varphi\, dx - \int\limits_{\mathbb{R}^N} h(x)|u|^{r-2}u\varphi\, dx$$

for all $\varphi \in X$, since $\langle \Psi_\lambda'(u_n), \varphi\rangle \to 0$ as $n \to \infty$ for all $\varphi \in X$. In conclusion, u is a critical point for Ψ_λ and so u is a non-negative entire solution of $(\mathscr{P}_\lambda)$. We claim that

$$\|u_n - u\| \to 0 \quad \text{as } n \to \infty. \tag{5.2}$$

First,

$$\mathcal{J}_w(n) = \int\limits_{\mathbb{R}^N} w(x)\big(\big|u_n^+\big|^{q-2}u_n^+ - \big|u^+\big|^{q-2}u^+\big)(u_n - u)\, dx \to 0 \tag{5.3}$$

as $n \to \infty$. Indeed, $u_n \to u$ in $L^q(\mathbb{R}^N, w)$ as stated in (4.8). Consequently, $u_n^+ \to u^+$ in $L^q(\mathbb{R}^N, w)$ and so also $|u_n^+|^{q-2}u_n^+ \to |u^+|^{q-2}u^+$ in $L^{q'}(\mathbb{R}^N, w)$ by Proposition A.8(ii) of [4]. Applying Hölder's inequality, we get

$$\big|\mathcal{J}_w(n)\big| \leqslant \big\|\big|u_n^+\big|^{q-2}u_n^+ - \big|u^+\big|^{q-2}u^+\big\|_{q',w}\|u_n - u\|_{q,w} \to 0,$$

as $n \to \infty$. This completes the proof of (5.3).

Put $\mathcal{R}_n = \mathcal{I}_1(n) + \mathcal{I}_2(n) + \mathcal{I}_3(n)$, where

$$\mathcal{I}_1(n) = [u_n - u]_s^2 \geqslant 0, \qquad \mathcal{I}_2(n) = \|u_n - u\|_{2,a}^2 \geqslant 0,$$
$$\mathcal{I}_3(n) = \int\limits_{\mathbb{R}^N} h(x)\big(|u_n|^{r-2}u_n - |u|^{r-2}u\big)(u_n - u)\, dx \geqslant 0.$$

Clearly $\langle \Psi_\lambda'(u_n) - \Psi_\lambda'(u), u_n - u\rangle \to 0$ as $n \to \infty$, since $u_n \rightharpoonup u$ in X and $\Psi_\lambda'(u_n) \to 0$ in X' as $n \to \infty$. Hence, by (5.3)

$$\mathcal{R}_n = \big\langle \Psi_\lambda'(u_n) - \Psi_\lambda'(u), u_n - u\big\rangle + \lambda\mathcal{J}_w(n) = o(1)$$

as $n \to \infty$, so that

$$\|u_n - u\|_E^2 \leqslant \mathcal{I}_1(n) + \frac{1}{\kappa}\mathcal{I}_2(n) = o(1) \tag{5.4}$$

as $n \to \infty$ by (1.2), and also as $n \to \infty$

$$\|u_n - u\|_{r,h}^r \leqslant k_r \mathcal{I}_3(n) = o(1), \tag{5.5}$$

thanks to Simon's inequality $|\xi - \xi_0|^r \leqslant k_r(|\xi|^{r-2}\xi - |\xi_0|^{r-2}\xi_0) \cdot (\xi - \xi_0)$ valid for all $\xi, \xi_0 \in \mathbb{R}$, being $r > 2$. Clearly (5.4) and (5.5) imply the claim (5.2).

Since $u_n \to u$ in X and $\Psi_\lambda \in C^1(X)$, we have $\Psi_\lambda(u) = c = \lim_{n\to\infty} \Psi_\lambda(u_n)$. Therefore, u is a second independent nontrivial non-negative entire solution of $(\mathscr{P}_\lambda)$, with $\Psi_\lambda(u) = c > 0 > \Psi_\lambda(e)$. This concludes the proof. □

Acknowledgments

The authors thank the referee for providing Refs. [13,14,19,22,27,28,32] and for suggesting improvements of the presentation. This paper has been supported by the MIUR project *Metodi Variazionali ed Equazioni Differenziali alle Derivate Parziali Non Lineari*.

Appendix A

In the following proposition we show that the Banach space X defined in the Introduction is reflexive. We insert this result for completeness of the presentation, even if it could be fairly foreseeable.

Proposition A.1. *The Banach space $(X, \|\cdot\|)$ is reflexive.*

Proof. We follow essentially the proof of Proposition A.11 of [4]. The product space $Y = E \times L^r(\mathbb{R}^N, h)$, endowed with the norm $\|u\|_Y = \|u\|_E + \|u\|_{r,h}$, is a reflexive Banach space by Theorem 1.22(ii) of [1], since E is a Hilbert space and $L^r(\mathbb{R}^N, h)$ is a uniformly convex Banach space by Proposition A.6 of [4].

The operator $T : (X, \|\cdot\|_Y) \to (Y, \|\cdot\|_Y)$, $T(u) = (u, u)$, is well defined, linear and isometric. Therefore, $T(X)$ is a closed subspace of the reflexive space Y, and so $T(X)$ is reflexive by Theorem 1.21(ii) of [1]. Consequently, $(X, \|\cdot\|_Y)$ is reflexive, being isomorphic to a reflexive Banach space. Finally, we conclude that also $(X, \|\cdot\|)$ is reflexive, because reflexivity is preserved under equivalent norms. □

We present the next result for the main solution space X, even if it continues to hold also for the larger space E.

Proposition A.2. *Let $(u_n)_n$, $u \in X$ be such that $u_n \rightharpoonup u$ in X. Then, up to a subsequence, $u_n \to u$ a.e. in $\mathbb{R}^N$.*

Proof. Let $(u_n)_n$ and u be as in the statement. Then, $u_n \to u$ as $n \to \infty$ in $L^p(B_R)$ for all $R > 0$ and $p \in [1, 2^*)$ by Lemma 2.1. In particular, in correspondence to $R = 1$ we find a subsequence $(u_{1,n})_n$ of $(u_n)_n$ such that $u_{1,n} \to u$ a.e. in B_1. Clearly $u_{1,n} \rightharpoonup u$ in X and so, in correspondence to $R = 2$, there exists a subsequence $(u_{2,n})_n$ of $(u_{1,n})_n$ such that $u_{2,n} \to u$ a.e. in B_2, and so on. The diagonal subsequence $(u_{n,n})_n$ of $(u_n)_n$, constructed by induction, converges to u a.e. in $\mathbb{R}^N$ as $n \to \infty$. □

The next three lemmas provide regularity properties for the main functionals involved in Φ_λ and Ψ_λ, defined in Section 2.

Lemma A.3. *The functional $\Phi_a : X \to \mathbb{R}$, $\Phi_a(u) = \frac{1}{2}\|u\|_{2,a}^2$, is convex, of class C^1 and weakly lower semicontinuous in X. Moreover, if $(u_n)_n$, $u \in X$ and $u_n \rightharpoonup u$ in X, then $\Phi_a'(u_n) \overset{*}{\rightharpoonup} \Phi_a'(u)$ in X'.*

Proof. The convexity of Φ_a is obvious. The embeddings $X \hookrightarrow E \hookrightarrow L^2(\mathbb{R}^N, \nu) \hookrightarrow L^2(\mathbb{R}^N, a)$ are continuous by (1.2), with $\|u\|_{2,a} \leqslant \|u\|$ for all $u \in X$. Hence the functional Φ_a is continuous in X. Consequently, Φ_a is weakly lower semicontinuous by Corollary 3.9 of [9].

Moreover, Φ_a is Gâteaux-differentiable in X and for all $u, \varphi \in X$ we have

$$\langle \Phi_a'(u), \varphi \rangle = \int_{\mathbb{R}^N} a(x) u \varphi \, dx.$$

Now, let $(u_n)_n$, $u \in X$ be such that $u_n \rightharpoonup u$ in X as $n \to \infty$. The embedding $X \hookrightarrow L^2(\mathbb{R}^N, a)$ implies that $u_n \rightharpoonup u$ in $L^2(\mathbb{R}^N, a)$. Thus, for all $\varphi \in X$

$$\int_{\mathbb{R}^N} a(x) u_n \varphi \, dx \to \int_{\mathbb{R}^N} a(x) u \varphi \, dx \tag{A.1}$$

as $n \to \infty$, that is $\langle \Phi_a'(u_n), \varphi \rangle \to \langle \Phi_a'(u), \varphi \rangle$, and so $\Phi_a'(u_n) \overset{*}{\rightharpoonup} \Phi_a'(u)$ in X', as claimed.

Let us prove that $\Phi_a \in C^1(X)$. Fix $(u_n)_n$, $u \in X$, with $u_n \to u$ in X. Hence $u_n \to u$ in $L^2(\mathbb{R}^N, a)$. Therefore, for all $\varphi \in X$, with $\|\varphi\| = 1$,

$$\left| \langle \Phi_a'(u_n) - \Phi_a'(u), \varphi \rangle \right| \leqslant \|u_n - u\|_{2,a} \|\varphi\|_{2,a} \leqslant \|u_n - u\|_{2,a},$$

since $\|\varphi\|_{2,a} \leqslant \|\varphi\|_{2,\nu} \leqslant \|\varphi\|$ for all $\varphi \in X$ by (1.2). Therefore,

$$\left\| \Phi_a'(u_n) - \Phi_a'(u) \right\|_{X'} \leqslant \|u_n - u\|_{2,a} \to 0$$

as $n \to \infty$. In conclusion, Φ_a is of class $C^1(X)$. □

Lemma A.4. *The functional $\Phi_w : X \to \mathbb{R}$, $\Phi_w(u) = \frac{1}{q}\|u\|_{q,w}^q$, is convex, of class C^1 and weakly continuous in X. Moreover, if $(u_n)_n$, $u \in X$ and $u_n \rightharpoonup u$ in X, then $\Phi_w'(u_n) \to \Phi_w'(u)$ in X'.*

Finally, the same properties hold for the functional $\Phi_w^+(u) = \frac{1}{q}\|u^+\|_{q,w}^q$.

Proof. First note that Φ_w is convex since $q > 2$. Moreover, by Lemma 2.2 and Theorem 3.10 of [9], we also have that Φ_w is weakly continuous, so that in particular Φ_w is continuous in X. Furthermore, Φ_w is Gâteaux-differentiable in X and for all $u, \varphi \in X$

$$\langle \Phi_w'(u), \varphi \rangle = \int_{\mathbb{R}^N} w(x) |u|^{q-2} u \varphi \, dx.$$

Now, let $(u_n)_n$, $u \in X$ be such that $u_n \rightharpoonup u$ in X and fix $\varphi \in X$, with $\|\varphi\| = 1$. By Lemma 2.2 and Proposition A.8(ii) of [4], it follows that $v_n = |u_n|^{q-2} u_n \to v = |u|^{q-2} u$ in $L^{q'}(\mathbb{R}^N, w)$. Therefore,

$$\left| \langle \Phi_w'(u_n) - \Phi_w'(u), \varphi \rangle \right| \leqslant \|v_n - v\|_{q',w} \|\varphi\|_{q,w} \leqslant \mathfrak{C}_w \|v_n - v\|_{q',w}$$

by (2.2). Hence,

$$\left\| \Phi_w'(u_n) - \Phi_w'(u) \right\|_{X'} \leqslant \mathfrak{C}_w \|v_n - v\|_{q',w},$$

that is $\Phi_w'(u_n) \to \Phi_w'(u)$ in X'. In particular, this shows that Φ_w is of class $C^1(X)$ and completes the proof of the first part of the lemma.

The last part is a direct consequence of the fact that if $u_n \rightharpoonup u$ in X, then $u_n \to u$ in $L^q(\mathbb{R}^N, w)$, and so $u_n^+ \to u^+$ in $L^q(\mathbb{R}^N, w)$, being $|u_n^+ - u^+| \leqslant |u_n - u|$ a.e. in $\mathbb{R}^N$. $\square$

Lemmas A.3 and A.4 continue to hold when X is replaced by E. Indeed, $E \hookrightarrow L^2(\mathbb{R}^N, a)$ by (1.2) and $E \hookrightarrow\hookrightarrow L^q(\mathbb{R}^N, w)$ by Lemma 2.2, so that all the functionals are well defined in E.

Lemma A.5. *The functional $\Phi_h : X \to \mathbb{R}$, $\Phi_h(u) = \frac{1}{r}\|u\|_{r,h}^r$ is convex, of class C^1 and weakly lower semicontinuous in X. Moreover, if $(u_n)_n$, $u \in X$ and $u_n \rightharpoonup u$ in X as $n \to \infty$, then $\Phi_h'(u_n) \overset{*}{\rightharpoonup} \Phi_h'(u)$ in X'.*

Proof. The convexity of Φ_h is obvious, being $r > 2$, while the continuity follows from the continuity of the embedding $X \hookrightarrow L^r(\mathbb{R}^N, h)$. Hence Φ_h is weakly lower semicontinuous in X by Corollary 3.9 of [9].

On the other hand, Φ_h is Gâteaux-differentiable in X and for all $u, \varphi \in X$

$$\langle \Phi_h'(u), \varphi \rangle = \int_{\mathbb{R}^N} h(x)|u|^{r-2} u\varphi \, dx.$$

Let $(u_n)_n$, $u \in X$ be such that $u_n \to u$ in X. Then, $u_n \to u$ in $L^r(\mathbb{R}^N, h)$, and so $v_n = |u_n|^{r-2}u_n \to v = |u|^{r-2}u$ in $L^{r'}(\mathbb{R}^N, h)$ by Proposition A.8(ii) of [4]. Therefore,

$$\left\| \Phi_h'(u_n) - \Phi_h'(u) \right\|_{X'} \leqslant \sup_{\substack{\varphi \in X \\ \|\varphi\|=1}} \|v_n - v\|_{r',h} \cdot \|\varphi\|_{r,h} \leqslant \|v_n - v\|_{r',h} = o(1)$$

as $n \to \infty$. This gives the C^1 regularity of Φ_h.

Suppose now that $u_n \rightharpoonup u$ in X. Fix a subsequence $(v_{n_k})_k$ of the sequence $n \mapsto v_n = |u_n|^{r-2}u_n$. Of course $u_{n_k} \rightharpoonup u$ in X and by Proposition A.2 there exists a further subsequence $(u_{n_{k_j}})_j$ such that $u_{n_{k_j}} \to u$ a.e. in $\mathbb{R}^N$. Thus $v_{n_{k_j}} \to v = |u|^{r-2}u$ a.e. in $\mathbb{R}^N$. On the other hand, $(v_{n_{k_j}})_j$ is bounded in $L^{r'}(\mathbb{R}^N, h)$, since $\|v_{n_{k_j}}\|_{r',h}^{r'} = \|u_{n_{k_j}}\|_{r,h}^r$ and $(u_{n_{k_j}})_j$ is bounded in $L^r(\mathbb{R}^N, h)$. Consequently, $v_{n_{k_j}} \rightharpoonup v$ in $L^{r'}(\mathbb{R}^N, h)$ by Proposition A.8(i) of [4]. In conclusion, due to the arbitrariness of $(v_{n_k})_k$, the entire sequence $v_n \rightharpoonup v$ in $L^{r'}(\mathbb{R}^N, h)$ as $n \to \infty$. In particular for all $\varphi \in X$

$$\int_{\mathbb{R}^N} h(x)|u_n|^{r-2} u_n \varphi \, dx \to \int_{\mathbb{R}^N} h(x)|u|^{r-2} u\varphi \, dx \tag{A.2}$$

as $n \to \infty$. This gives the claim and completes the proof. $\square$

Appendix B

In this section we present the few changes we need to prove Theorem 1.1, when $\mathscr{L}_K u$ replaces $(-\Delta)^s u$ in $(\mathscr{P}_\lambda)$ and $K : \mathbb{R}^N \setminus \{0\} \to \mathbb{R}^+$ satisfies the main properties (k_1)–(k_3) of the Introduction.

By (k_1) for all $\varphi \in C_0^\infty(\mathbb{R}^N)$ the function

$$(x, y) \mapsto \left[u(x) - u(y)\right] \cdot \sqrt{K(x-y)} \in L^2\left(\mathbb{R}^{2N}\right).$$

Let us denote by $D_K^s(\mathbb{R}^N)$ the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the Hilbertian norm

$$[u]_{s,K} = \left(\iint_{\mathbb{R}^{2N}} \left|u(x) - u(y)\right|^2 K(x-y) \, dx \, dy \right)^{1/2},$$

induced by the inner product

$$\langle u, v\rangle_{s,K} = \iint_{\mathbb{R}^{2N}} \big[u(x) - u(y)\big] \cdot \big[v(x) - v(y)\big] \cdot K(x-y)\,dx\,dy.$$

Clearly the embedding $D^s_K(\mathbb{R}^N) \hookrightarrow D^s(\mathbb{R}^N)$ is continuous, being

$$[u]_s \leqslant \gamma^{-1/2}[u]_{s,K} \quad \text{for all } u \in D^s_K\big(\mathbb{R}^N\big),$$

by (k_2). Hence (2.1) holds for all $u \in D^s_K(\mathbb{R}^N)$.

Let E_K denote the completion of $C^\infty_0(\mathbb{R}^N)$ with respect to the Hilbertian norm

$$\|u\|_{E,K} = \big([u]^2_{s,K} + \|u\|^2_{2,\nu}\big)^{1/2},$$

induced by the inner product $\langle u, v\rangle_{E,K} = \langle u, v\rangle_{s,K} + \langle u, v\rangle_\nu$. Finally, X_K is the completion of $C^\infty_0(\mathbb{R}^N)$ with respect to the norm

$$\|u\|_K = \big(\|u\|^2_{E,K} + \|u\|^2_{r,h}\big)^{1/2},$$

and X_K is a reflexive Banach space, as it can be shown adapting the proof of Proposition A.1.

By the above remarks and Lemma 2.1 it is clear that the embeddings $X_K \hookrightarrow E_K \hookrightarrow D^s_K(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$ are continuous, with $[u]_{s,K} \leqslant \|u\|_{E_K}$ for all $u \in E_K$ and $\|u\|_{E_K} \leqslant \|u\|_K$ for all $u \in X_K$, and that for any $R > 0$ and $p \in [1, 2^*)$ the embeddings $E_K \hookrightarrow\hookrightarrow L^p(B_R)$ and $X_K \hookrightarrow\hookrightarrow L^p(B_R)$ are compact. Similarly, by Lemma 2.2 the embeddings $E_K \hookrightarrow\hookrightarrow L^q(\mathbb{R}^N, w)$ and $X_K \hookrightarrow\hookrightarrow L^q(\mathbb{R}^N, w)$ are compact, and

$$[u]^2_{s,K} + \|u\|^2_{2,a} \geqslant \kappa \|u\|^2_{E_K}$$

for all $u \in E_K$, where κ is given in (1.2).

A (*weak*) *entire solution* of

$$\mathscr{L}_K u + a(x)u = \lambda w(x)|u|^{q-2}u - h(x)|u|^{r-2}u \quad \text{in } \mathbb{R}^N \tag{$\mathcal{L}_\lambda$}$$

is a function $u \in X_K$ such that

$$\langle u, \varphi\rangle_{s,K} + \int_{\mathbb{R}^N} a(x)u\varphi\,dx = \lambda \int_{\mathbb{R}^N} w(x)|u|^{q-2}u\varphi\,dx - \int_{\mathbb{R}^N} h(x)|u|^{r-2}u\varphi\,dx$$

for all $\varphi \in X_K$. Actually the entire solutions of $(\mathcal{L}_\lambda)$ correspond to the critical points of the energy functional $\mathscr{I}_\lambda : X_K \to \mathbb{R}$, defined by

$$\mathscr{I}_\lambda(u) = \frac{1}{2}[u]^2_{s,K} + \frac{1}{2}\|u\|^2_{2,a} - \frac{\lambda}{q}\|u\|^q_{q,w} + \frac{1}{r}\|u\|^r_{r,h}.$$

Similarly, we can prove that non-negative entire solutions of $(\mathcal{L}_\lambda)$ are exactly the critical points of the functional

$$\mathscr{J}_\lambda(u) = \frac{1}{2}[u]^2_{s,K} + \frac{1}{2}\|u\|^2_{2,a} - \frac{\lambda}{q}\|u^+\|^q_{q,w} + \frac{1}{r}\|u\|^r_{r,h},$$

well-defined for all $u \in X_K$, just adapting the previous argument of Section 2, using now (k_3).

Following the proof of Lemma 2.3 it is clear that if $u \in X_K \setminus \{0\}$ and $\lambda \in \mathbb{R}$ satisfy

$$[u]^2_{s,K} + \|u\|^2_{2,a} + \|u\|^r_{r,h} = \lambda \|u\|^q_{q,w},$$

then $\lambda > 0$ and (2.6) continues to hold. Therefore, if $(\mathcal{L}_\lambda)$ admits a *nontrivial* entire solution $u \in X_K$, then $\lambda \geqslant \lambda_0 = (c_1/c_2)^{2(r-q)(q-2)/q(r-2)} > 0$, where now in c_1 and c_2 the constant $\mathfrak{C}_w = \gamma^{-1/2} C_{2^*} \|w\|^{1/q}_{\wp} > 0$. The crucial numbers

$$\lambda^*_K = \sup\{\lambda > 0\colon\ (\mathcal{L}_\mu)\ \textit{admits only the trivial solution for all}\ \mu < \lambda\},$$
$$\lambda^*_{\mathscr{J}_\lambda} = \sup\{\lambda > 0\colon\ (\mathcal{L}_\mu)\ \textit{admits no nontrivial non-negative solution for all}\ \mu < \lambda\}$$

are well defined and $\lambda^*_{\mathscr{J}_\lambda} \geqslant \lambda^*_K \geqslant \lambda_0 > 0$.

Lemmas 3.1–3.4, Proposition A.2, Lemmas A.3–A.5 and Lemma 4.1 continue to hold for $\mathscr{I}_\lambda$, $\mathscr{J}_\lambda$ and X_K in place of Φ_λ, Ψ_λ and X. Clearly, now

$$\bar{\lambda}_K = \inf_{\substack{u \in X_K \\ \|u\|_{q,w}=1}} \left\{\frac{q}{2}[u]^2_{s,K} + \frac{q}{2}\|u\|^2_{2,a} + \frac{q}{r}\|u\|^r_{r,h}\right\} > 0,$$
$$\lambda^{**}_K = \inf\{\lambda > 0\colon\ (\mathcal{L}_\lambda)\ \textit{admits a nontrivial non-negative entire solution}\},$$

and $\lambda^{**}_K \leqslant \bar{\lambda}_K$.

The main proof of the fact that for any $\lambda > \lambda^{**}_K$ Eq. $(\mathcal{L}_\lambda)$ admits a nontrivial non-negative entire solution $u_\lambda \in X_K$ follows word by word, with obvious changes in notation, from the proof of Theorem 4.2. Indeed, the key inequalities involving

$$\mathscr{U}_\varepsilon(x, y) = [u(x) - u(y)] \cdot [\varphi_\varepsilon(x) - \varphi_\varepsilon(y)] \cdot K(x - y),$$
$$\mathscr{U}(x, y) = [u(x) - u(y)] \cdot [\varphi(x) - \varphi(y)] \cdot K(x - y)$$

follow by (k_3) and the fact that $\mathscr{U} \in L^1(\mathbb{R}^{2N})$, being $X_K \hookrightarrow D^s_K(\mathbb{R}^N)$. Moreover, the proof that $(\mathcal{L}_{\lambda^{**}_K})$ admits a nontrivial non-negative entire solution in X_K can proceed as in Theorem 4.3, with obvious changes.

Finally, it goes without saying that the entire Section 5 continues to hold when $\mathscr{J}_\lambda$, X_K and E_K replace Ψ_λ, X and E.

Therefore, under the structural assumptions of the Introduction and (k_1)–(k_3) *there exist* λ^*_K, λ^{**}_K *and* $\bar{\lambda}_K$, *with* $0 < \lambda^*_K \leqslant \lambda^{**}_K \leqslant \bar{\lambda}_K$, *such that* $(\mathcal{L}_\lambda)$ *admits*

(i) *only the trivial solution if* $\lambda < \lambda^*_K$;
(ii) *a nontrivial non-negative entire solution if and only if* $\lambda \geqslant \lambda^{**}_K$;
(iii) *at least two nontrivial non-negative entire solutions if* $\lambda > \bar{\lambda}_K$.

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