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ON THE EXISTENCE OF STATIONARY SOLUTIONS FOR HIGHER ORDER p–KIRCHHOFF PROBLEMS

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In this paper we establish the existence of two nontrivial weak solutions of *possibly degenerate* nonlinear eigenvalue problems involving the p–polyharmonic Kirchhoff operator in bounded domains. The p–polyharmonic operators Δ_p^L were recently introduced in [1] for all orders L and independently, in the same volume of the journal, in [2] only for L even. In Section 3 the results are then extended to *non–degenerate* $p(x)$–polyharmonic Kirchhoff operators. The main tool of the paper is a three critical points theorem given in [3]. Several useful properties of the underlying functional solution space $[W_0^{L,p}(\Omega)]^d$, endowed with the natural norm arising from the variational structure of the problem, are also proved both in the homogeneous case $p \equiv Const.$ and in the non–homogeneous case $p = p(x)$. In the latter some sufficient conditions on the variable exponent p are given to prove the positivity of the infimum λ_1 of the Rayleigh quotient for the $p(x)$–polyharmonic operator $\Delta_{p(x)}^L$.

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1. Introduction

The three critical points theorems due to *Pucci* and *Serrin* [4,5] and to *Ricceri* [6] have been extensively applied in several problems involving differential equations with a real parameter. For applications we refer to the recent book by *Kristály*, *Rădulescu* and *Varga* [7] and the references therein. *Arcoya* and *Carmona* extended in [8, Theorem 3.4] the results of *Pucci* and *Serrin* to a wide class of continuous functionals not necessarily differentiable. A slightly modified version of [8, Theorem 3.4] has been recently established in [3, Theorem 2.1], and applied to an eigenvalue problem involving a divergence type operator. In this paper we use Theorem 2.1 of [3] in order to extend the previous results to non–local higher order problems.

First, we consider the eigenvalue p–Kirchhoff Dirichlet problem

$$\begin{cases} M(\|u\|^p)\Delta_p^L u = \lambda\{\gamma\|u\|_{p,w}^{p(\gamma-1)}w(x)|u|^{p-2}u + f(x,u)\} & \text{in } \Omega, \\ D^\alpha u_k\big|_{\partial\Omega} = 0 \qquad \text{for all } \alpha, \text{ with } |\alpha| \le L-1, \text{ and all } k = 1,\dots,d, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain, $n \ge 1$, $u = (u_1,\dots,u_d) = u(x)$, $d \ge 1$, $p > 1$, $L = 1,2,\dots$, $\lambda \in \mathbb{R}$, α is a multi–index, $\gamma \in [1, p_L^*/p)$ and p_L^* is the critical Sobolev exponent

$$p_L^* = \begin{cases} \dfrac{np}{n-Lp}, & \text{if} \quad n > Lp, \\ \infty, & \text{if } 1 \le n \le Lp. \end{cases} \tag{1.2}$$

The vectorial p–polyharmonic operator Δ_p^L is defined by

$$\Delta_p^L\varphi = \begin{cases} \mathcal{D}_L\big(|\mathcal{D}_L\varphi|^{p-2}\mathcal{D}_L\varphi\big), & \text{if } L = 2j, \\ -\operatorname{div}\left\{\Delta^{j-1}\left(|\mathcal{D}_L\varphi|^{p-2}\mathcal{D}_L\varphi\right)\right\}, & \text{if } L = 2j-1, \end{cases} \quad \text{for } j = 1,2,\dots,$$

for all $\varphi = (\varphi_1,\dots,\varphi_d) \in [C_0^\infty(\Omega)]^d$, where $\mathcal{D}_L$ denotes the vectorial operator

$$\mathcal{D}_L\varphi = \begin{cases} (\Delta^j\varphi_1,\dots,\Delta^j\varphi_d), & \text{if } L = 2j, \\ (D\Delta^{j-1}\varphi_1,\dots,D\Delta^{j-1}\varphi_d), & \text{if } L = 2j-1, \end{cases} \quad \text{for } j = 1,2,\dots. \tag{1.3}$$

For all $x \in \Omega$ the vector $\mathcal{D}_L\varphi(x)$ has dimension d if L is even or dn if L is odd and, in both cases, the dimension is simply denoted by N. The vectorial p–polyharmonic operator Δ_p^L in the weak sense is

$$\langle \Delta_p^L u, \varphi\rangle = \int_\Omega |\mathcal{D}_L u|^{p-2}\mathcal{D}_L u \cdot \mathcal{D}_L\varphi\, dx$$

for all u, $\varphi \in [W_0^{L,p}(\Omega)]^d$.

When $d = 1$ the scalar p–polyharmonic operator Δ_p^L was first introduced in [1] for all $L \ge 1$ and $p > 1$. In the scalar case Δ_p^2 is exactly the well–known *p–biharmonic operator* $\Delta_p^2\psi = \Delta(|\Delta\psi|^{p-2}\Delta\psi)$ for all $\psi \in C_0^\infty(\Omega)$ defined in the pioneering paper [9] by *Kratochvl* and *Nečăs*; see also [10] by *Drábek* and *Ôtani*. In [2] *Lubyshev* proved the existence of multiple solutions of a nonlinear Dirichlet problem governed by the scalar operator Δ_p^L only for L even.

In the 2–dimensional scalar case (1.1) arises from the theory of thin plates and describes the deflection $u = u(x_1, x_2)$ of the middle surface of a p–power–like elastic isotropic flat plate of uniform thickness, with non–local flexural rigidity of the plate $M(\|u\|^p)$ depending continuously on $\|u\|^p$ of the deflection u and subject to nonlinear source forces. The coordinates (x_1, x_2) are taken in the plane $x_3 = 0$ of the middle surface of the plate before bending. For other scalar problems modeled by (1.1) we refer to the Introduction of [1]. For more standard polyharmonic problems we mention the recent monograph [11].

Problem (1.1) is a nonlinear perturbation of the natural eigenvalue problem associated to the non–local higher order operator $M(\|u\|^p)\Delta_p^L u$. The perturbation $f:\Omega\times\mathbb{R}^d\to\mathbb{R}^d$ is a *Carathéodory function*, with growth at infinity q, $1<q<p$. The main assumption $(\mathcal{F})$ on f is stated in Section 3.1. The weight w is *positive a.e. in* Ω *and*

$$w\in L^{\varpi}(\Omega),\qquad \varpi>\frac{n}{n-\gamma[n-Lp]^+}. \tag{1.4}$$

Restriction (1.4) is meaningful, being $\gamma\in[1,p_L^*/p)$.

The *Kirchhoff function* $M:\mathbb{R}_0^+\to\mathbb{R}_0^+$ is assumed to verify the general structural assumption

$(\mathcal{M})$ *M is continuous, non–decreasing and there exists* $s>0$ *such that*

$$s\gamma\tau^\gamma\le\tau M(\tau)\quad\textit{for all } \tau\in\mathbb{R}^+.$$

From now on, we denote by $\mathscr{M}(\tau)=\displaystyle\int_0^\tau M(z)dz$ for all $\tau\in\mathbb{R}_0^+$. Problem (1.1) is called *degenerate* if $M(0)=0$, otherwise, if $M(0)>0$, it is *non–degenerate.*

The standard Kirchhoff function introduced in [12] is

$$M(\tau)=a+b\gamma\tau^{\gamma-1},\quad a,b\ge0,\quad a+b>0,\quad\text{with}$$
$$\gamma\begin{cases}\in(1,p_L^*/p), & \text{if } b>0,\\ =1, & \text{if } b=0,\end{cases}\qquad s=\begin{cases}b, & \text{if } b>0,\\ a, & \text{if } b=0,\end{cases}$$

which clearly verifies condition $(\mathcal{M})$. For such M's (1.1) is *degenerate* if $a=0$ and $b>0$, and *non–degenerate* when $a>0$ and $b\ge0$. Finally, when $a>0$ and $b=0$, the Kirchhoff function M is simply a constant and (1.1) reduces to a local quasilinear elliptic Dirichlet problem.

In Theorem 2.1 we determine precisely the interval of λ's for which (1.1) has only the trivial solution and then, using the three critical points Theorem 2.1 of [3], the interval of λ's for which (1.1) admits at least two nontrivial solutions. The main difficult point of Sections 2.3 and 2.4 is *to cover the more delicate degenerate case*, in which compactness properties are harder to handle. For this reason, even the most recent papers on stationary problems cover only the non–degenerate case, where $\gamma=1$ in $(\mathcal{M})$, that is when $M(\tau)\ge s>0$ for all $\tau\in\mathbb{R}_0^+$; see e.g. [13,14,15]. The efforts in treating the degenerate case require a special care and a deeper analysis, as the main proof of Lemma 2.3 shows. A preliminary study of Δ_p^L only when $d=1$ has been first developed in [1], where a possibly degenerate scalar stationary p–polyharmonic Kirchhoff Dirichlet problem has been considered.

In the higher order vectorial setting several different norms are available for the solution functional space $[W_0^{L,p}(\Omega)]^d$. In Section 2.1 we prove the equivalence between the standard Sobolev norm and the norm $\|u\|=\|\,|\mathcal{D}_Lu|_N\,\|_p$, which is the natural norm arising from the variational structure of problem (1.1). The proof of the equivalence is based on the *Poincaré* and *Caldéron–Zygmund* inequalities and relies on Proposition A.1 proved in [1] when $d=1$. However, the space $([W_0^{L,p}(\Omega)]^d,\|\cdot\|)$

is uniformly convex, as shown in Proposition Appendix A.2 by a useful inequality given in Lemma A.1 of [16].

We conclude Section 2.4 with the easier problem (2.27), which is the main model first treated in [17] when $M \equiv 1$, $L = d = 1$ and $p \geq 2$, and in which the right–hand side of the system presents only the term $\lambda f(x, u)$. Theorem 2.2, an analogue of Theorem 2.1, is proved under a simpler and more direct condition on f and without the use of the first eigenfunction of Δ_p^L.

In Section 2.5 we assume that $\gamma = 1$, that is we deal with the *non–degenerate* case of (1.1), being $M(\tau) \geq s > 0$ for all $\tau \in \mathbb{R}_0^+$ by $(\mathcal{M})$. Hence, Section 2.5 is devoted to the study of the special higher order p–Kirchhoff problem

$$\begin{cases} M(\|u\|^p)\Delta_p^L u = \lambda\{w(x)|u|^{p-2}u + f(x,u)\} & \text{in } \Omega, \\ D^\alpha u_k\big|_{\partial\Omega} = 0 & \text{for all } \alpha, \text{ with } |\alpha| \leq L-1, \text{ and all } k = 1, \dots, d, \end{cases} \tag{1.5}$$

where here w satisfies (1.4), with $\gamma = 1$.

In Section 3 we extend the results of Section 2.5 to the $p(x)$–polyharmonic Kirchhoff problem

$$\begin{cases} M(\mathscr{I}_L(u))\Delta_{p(x)}^L u = \lambda\{w(x)|u|^{p(x)-2}u + f(x,u)\} & \text{in } \Omega, \\ D^\alpha u_k\big|_{\partial\Omega} = 0 & \text{for all } \alpha, \text{ with } |\alpha| \leq L-1, \text{ and all } k = 1, \dots, d, \end{cases} \tag{1.6}$$

where now $\Omega \subset \mathbb{R}^n$ *is a bounded domain with Lipschitz boundary* and

$$\mathscr{I}_L(u) = \int_\Omega \frac{|\mathcal{D}_L u|^{p(x)}}{p(x)} dx \tag{1.7}$$

is the Dirichlet functional associated to the weak form of $\Delta_{p(x)}^L$, that is related to

$$\langle \Delta_{p(x)}^L u, \varphi \rangle = \int_\Omega |\mathcal{D}_L u|^{p(x)-2} \mathcal{D}_L u \cdot \mathcal{D}_L \varphi dx$$

for all u, $\varphi \in [W_0^{L,p(\cdot)}(\Omega)]^d$. The solution functional space is the vector–valued variable exponent Sobolev space $[W_0^{L,p(\cdot)}(\Omega)]^d$, which in the scalar case $d = 1$ has been extensively studied in the last two decades, see [18]–[22]. Indeed, the variable exponent Lebesgue and Sobolev spaces arouse a great interest not only for the mathematical curiosity, but also for concrete applications. For instance, in models where linear elasticity (Hooke's law) is replaced by $p(x)$–power–like elasticity. Problem (1.6) can be used in modeling steady electrorheological fluids (that is fluids whose mechanical properties strongly depend on the applied electromagnetic field). See [23] and also [24] for more specific comments. The range of applications of electrorheological fluids is wide and includes vibration absorbers, engine mounts, earthquake–resistant buildings, clutches, etc.

However, the vectorial case does not seem to be so well–known, so that we present the main properties of $[W_0^{L,p(\cdot)}(\Omega)]^d$ in Section 3 and in Appendix B.

We require that *the variable exponent* p *is of a specific class* $C_+(\overline{\Omega})$ defined in Setion 3 and satisfies all the standard assumptions usually required in this setting.

For simplicity, we also *assume that*

$$\textit{either} \qquad \frac{n}{L} > p_+ = \max_{\overline{\Omega}} p \qquad \textit{or} \qquad \frac{n}{L} \le p_- = \min_{\overline{\Omega}} p.$$

The weight w is *positive a.e. in* Ω *and of class* $L^{\varpi}(\Omega)$*, with* $\varpi > n/(n-[n-Lp_-]^+)$. Furthermore, the most interesting case occurs when $p_- < p_+$, that is in the so–called *nonstandard growth condition of* (p_-, p_+) *type*; cf. [25]. The main reason why the $p(x)$–Laplace operators possess more complicated behavior is the fact that they are no longer homogeneous. Moreover, the first eigenvalue λ_1 of the $p(x)$–Laplace Dirichlet problem could be zero, see [26]. When $L = d = 1$, *Fan* et al. in [26], *Mihăilescu* et al. in [27] and then *Allegretto* in [28] give sufficient conditions on p in order to have $\lambda_1 > 0$. In Section 3, precisely in Propositions 3.1 and 3.2, we carry on a long discussion on the positivity of the first eigenvalue of the $p(x)$–polyharmonic Dirichlet problem when L, $d \ge 1$.

Finally, in Theorem 3.1 we determine precisely the interval of λ's for which problem (1.6) has only the trivial solution and then, using again Theorem 2.1 of [3], the interval of λ's for which (1.6) admits at least two nontrivial solutions.

2. The main result for (1.1)

2.1. *Preliminaries*

In the scalar setting, by $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{N}_0^n$ we denote a multi–index, with length $|\alpha| = \sum_{i=1}^n \alpha_i \le L$ and the corresponding partial differentiation

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} ... \partial x_n^{\alpha_n}}.$$

Throughout this section we assume that $1 < p < \infty$ and denote by $W_0^{L,p}(\Omega)$ the completion of $C_0^\infty(\Omega)$ with respect to the standard norm $\|\psi\|_{W^{L,p}(\Omega)} = \left(\sum_{|\alpha|\le L} \|D^\alpha \psi\|_p^p\right)^{1/p}$.

By the *Poincaré* and the *Caldéron–Zygmund* inequalities, Proposition A.1 in [1] shows that the standard norm $\|\cdot\|_{W^{L,p}(\Omega)}$ and the norm

$$\|\psi\|_{L,p} = \begin{cases} \|\Delta^j \psi\|_p, & \text{if } L = 2j, \\ \left(\sum_{i=1}^n \|\partial_{x_i} \Delta^{j-1} \psi\|_p^p\right)^{1/p}, & \text{if } L = 2j-1, \end{cases} \qquad j = 1, 2, \dots, \tag{2.1}$$

are equivalent in $W_0^{L,p}(\Omega)$.

As already stated in the Introduction, since we are *in the vectorial setting*, we denote by $\mathcal{D}_L$ the vectorial operator defined in (1.3). Hence, if $L = 1$, the operator $\mathcal{D}_1$ writes the pointwise Jacobian matrix of u, $Ju(x) \in \mathcal{M}_{d\times n}(\mathbb{R})$, as the dn–row vector $\mathcal{D}_1 u(x) \in \mathbb{R}^{dn}$. Furthermore, for all $L = 1, 2, \dots$, the norm

$$\|u\|_{d,L,p} = \left(\sum_{k=1}^d \|u_k\|_{L,p}^p\right)^{1/p}$$

in $[W_0^{L,p}(\Omega)]^d$ is equivalent to the standard norm

$$\|u\|_{[W^{L,p}(\Omega)]^d} = \left(\sum_{k=1}^{d} \|u_k\|^p_{W^{L,p}(\Omega)}\right)^{1/p},$$

as a direct consequence of Proposition A.1 in [1]. Moreover, $\left([W_0^{L,p}(\Omega)]^d, \|\cdot\|_{d,L,p}\right)$ is a uniformly convex Banach space, as shown in Proposition Appendix A.1. However, since we are interested in the variational problem (1.1), *from now on we endow the space* $[W_0^{L,p}(\Omega)]^d$ *with the norm*

$$\|u\| = \|\,|\mathcal{D}_L u|_N\,\|_p,$$

where $|\cdot|_N$ denotes the Euclidean norm in $\mathbb{R}^N$ and $N = d$ when L is even, while $N = dn$ when L is odd. As shown in Proposition Appendix A.2, also *the space* $\left([W_0^{L,p}(\Omega)]^d, \|\cdot\|\right)$ *is uniformly convex.* An easy calculation shows that the two norms $\|\cdot\|_{d,L,p}$ and $\|\cdot\|$ are equivalent in $[W_0^{L,p}(\Omega)]^d$. Indeed, for all $u \in [W_0^{L,p}(\Omega)]^d$

$$\min\{1,\, N^{\frac{1}{p}-\frac{1}{2}}\}\|u\| \le \|u\|_{d,L,p} \le \max\{1,\, N^{\frac{1}{p}-\frac{1}{2}}\}\|u\|.$$

In particular, the two norms coincide whenever either $p = 2$, or $N = 1$.

The Lebesgue spaces $[L^\sigma(\Omega)]^m$ and $[L^\sigma(\Omega,\omega)]^m$, where $\sigma \ge 1$, ω is any weight on Ω and $m \ge 1$ is any dimension, are endowed with the norms $\|\varphi\|_{m,\sigma} = \|\,|\varphi|_m\|_\sigma$ and $\|\varphi\|_{m,\sigma,\omega} = \|\,|\varphi|_m\|_{\sigma,\omega}$, respectively. When $m = 1$ the norm is denoted by $\|\varphi\|_{\sigma,\omega}$. The dot $\cdot$ indicates the inner product and $|\cdot|_m$ denotes the Euclidean norm in $\mathbb{R}^m$. In what follows, when the dimension is clear from the context, we drop the subscript m and denote the m–Euclidean norm simply by $|\cdot|$.

As already noted in the Introduction, the main assumption

$$1 \le \gamma < p_L^*/p \quad \text{implies that} \quad \varpi > n/(n - \gamma[n - Lp]^+) \ge 1,$$

by (1.4). When $n > Lp$ then

$$\varpi' < n/\gamma(n - Lp), \quad \text{that is} \quad \gamma p < p_L^*/\varpi', \tag{2.2}$$

this will be useful in the next lemma. For simplicity in notation, whenever the embedding operator $i : [W_0^{L,p}(\Omega)]^d \to [L^\sigma(\Omega,\omega)]^d$ is continuous, we denote by $\mathcal{S}_{d,\sigma,w} > 0$ the best constant such that $\|u\|_{d,\sigma,\omega} \le \mathcal{S}_{d,\sigma,\omega}\|u\|$ for all $u \in [W_0^{L,p}(\Omega)]^d$, that is $\mathcal{S}_{d,\sigma,w}$ is the operator norm of i. If $d = 1$ and $\omega \equiv 1$, we briefly write $\mathcal{S}_\sigma$. Furthermore, whenever $p_L^* = \infty$, the symbol p_L^*/ϖ' is again ∞.

Lemma 2.1. *The following embeddings hold.*

(i) $[W_0^{L,p}(\Omega)]^d \hookrightarrow\hookrightarrow [L^\sigma(\Omega,w)]^d$ *compactly, if* $\sigma \in [1,\gamma p]$.
(ii) $[W_0^{L,p}(\Omega)]^d \hookrightarrow [L^\sigma(\Omega,w)]^d$ *continuously, if* $\sigma \in (\gamma p, p_L^*/\varpi')$.

Proof. *(i)* The space $W_0^{L,p}(\Omega)$ is compactly embedded into $L^{\varpi'\sigma}(\Omega)$, being $\varpi'\sigma < p_L^*$ by (1.2) and (2.2). Similarly, $L^{\varpi'\sigma}(\Omega)$ is continuously embedded in $L^\sigma(\Omega,w)$

by Hölder's inequality and (1.4). Hence, $\Big([W_0^{L,p}(\Omega)]^d, \|\cdot\|\Big)$ is compactly embedded into $\big([L^\sigma(\Omega,w)]^d, \|\cdot\|_{d,\sigma,w}\big)$, being $\|\cdot\|$ equivalent to $\|\cdot\|_{d,L,p}$ in $[W_0^{L,p}(\Omega)]^d$ as observed above.

(ii) By Hölder's inequality, for all $\psi \in W_0^{L,p}(\Omega)$

$$\|\psi\|^\sigma_{\sigma,w} \le |\Omega|^{1/\wp}\|w\|_\varpi \|\psi\|^\sigma_{p_L^*} \le C\|\psi\|^\sigma,$$

where $C = \mathcal{S}^\sigma_{p_L^*}|\Omega|^{1/\varpi}\|w\|_\varpi$ and $\wp$ is the crucial exponent

$$\wp = \begin{cases} \dfrac{\varpi' p_L^*}{p_L^* - \sigma\varpi'}, & \text{if} \qquad n > Lp, \\ \varpi', & \text{if } 1 \le n \le Lp. \end{cases}$$

Clearly $\wp > 1$, being $\sigma < p_L^*/\varpi'$. The conclusion now follows as in (i). $\square$

Let us now turn to the main problem (1.1) and let

$$\lambda_1 = \inf_{\substack{u\in[W_0^{L,p}(\Omega)]^d \\ u\neq 0}} \frac{\|u\|^p}{\|u\|^p_{d,p,w}} \tag{2.3}$$

be *the first eigenvalue* of

$$\begin{cases} \Delta_p^L u = \lambda\, w(x)|u|^{p-2}u & \text{in } \Omega, \\ D^\alpha u_k\big|_{\partial\Omega} = 0 & \text{for all } \alpha, \text{ with } |\alpha| \le L-1, \text{ and all } k = 1,\dots,d. \end{cases}$$

Clearly, λ_1 is well defined since the embedding $[W_0^{L,p}(\Omega)]^d \hookrightarrow\hookrightarrow [L^p(\Omega,w)]^d$ is compact, as shown in Lemma 2.1–(i).

Proposition 2.1. *The infimum λ_1 in* (2.3) *is positive and attained at a certain function $u_1 \in [W_0^{L,p}(\Omega)]^d$, with $\|u_1\|_{d,p,w} = 1$.*

Proof. For any $u \in [W_0^{L,p}(\Omega)]^d$ define the functionals $\mathcal{I}(u) = \|u\|^p$ and $\mathcal{J}(u) = \|u\|^p_{d,p,w}$. Let $\lambda_0 = \inf\{\mathcal{I}(u)/\mathcal{J}(u) \,:\, u \in [W_0^{L,p}(\Omega)]^d \setminus \{0\},\ \|u\|_{d,p,w} \le 1\}$. Observe that $\mathcal{I}$ and $\mathcal{J}$ are continuously Fréchet differentiable and convex in $[W_0^{L,p}(\Omega)]^d$. Clearly $\mathcal{I}'(0) = \mathcal{J}'(0) = 0$. Moreover, $\mathcal{J}'(u) = 0$ implies $u = 0$. In particular, $\mathcal{I}$ and $\mathcal{J}$ are weakly lower semi–continuous on $[W_0^{L,p}(\Omega)]^d$. Actually, $\mathcal{J}$ is weakly sequentially continuous on $[W_0^{L,p}(\Omega)]^d$. Indeed, if $(u_k)_k \subset [W_0^{L,p}(\Omega)]^d$ and $u_k \rightharpoonup u$ in $[W_0^{L,p}(\Omega)]^d$, then $u_k \to u$ in $[L^p(\Omega,w)]^d$ by Lemma 2.1–(i). This implies at once that $\mathcal{J}(u_k) = \|u_k\|^p_{d,p,w} \to \|u\|^p_{d,p,w} = \mathcal{J}(u)$, as claimed.

Now, either $W = \{u \in [W_0^{L,p}(\Omega)]^d \,:\, \mathcal{J}(u) \le 1\}$ is bounded in $[W_0^{L,p}(\Omega)]^d$, or not. In the first case we are done, while in the latter $\mathcal{I}$ is coercive in W, being coercive in the reflexive Banach space $[W_0^{L,p}(\Omega)]^d$ (see Proposition Appendix A.2). Therefore, all the assumptions of Theorem 6.3.2 of [29] are fulfilled and λ_0 is attained at a point $u_1 \in [W_0^{L,p}(\Omega)]^d$, with $\|u_1\|_{d,p,w} = 1$. We claim now that $\lambda_0 = \lambda_1$. Indeed,

$$\begin{aligned}\lambda_1 &= \inf_{u \in [W_0^{L,p}(\Omega)]^d \setminus \{0\}} \left\| \frac{u}{\|u\|_{d,p,w}} \right\|^p = \inf_{\substack{u \in [W_0^{L,p}(\Omega)]^d \\ \|u\|_{d,p,w}=1}} \|u\|^p \\ &\geq \inf_{\substack{u \in [W_0^{L,p}(\Omega)]^d \\ 0<\|u\|_{d,p,w}\leq 1}} \frac{\|u\|^p}{\|u\|_{d,p,w}^p} = \lambda_0 \geq \lambda_1.\end{aligned}$$

Finally, $\lambda_1 = \|u_1\|^p > 0$. This concludes the proof. □

By Lemma 2.1–(i), there exists $c_{\gamma p} = \mathcal{S}_{d,\gamma p,w}^{\gamma p} > 0$ such that

$$\|u\|_{d,\gamma p,w}^{\gamma p} \leq c_{\gamma p} \|u\|^{\gamma p} \quad \text{for all } u \in [W_0^{L,p}(\Omega)]^d. \tag{2.4}$$

When $\gamma = 1$ in $(\mathcal{M})$ we have $c_p = 1/\lambda_1 > 0$ by (2.3).

2.2. *The perturbation f*

The nonlinearity f verifies the next condition.

$(\mathcal{F})$ *Let $f : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$, $f = f(x,v) \not\equiv 0$, be a Carathéodory function, which admits a potential $F : \Omega \times \mathbb{R}^d \to \mathbb{R}$, $f = D_v F$, with $F(x,0) = 0$ a.e. in Ω, satisfying the following properties.*

(a) *There exist $q \in (1, \gamma p)$ and $C_f > 0$ such that*

$$|f(x,v)| \leq C_f w(x)\big(1 + |v|^{q-1}\big) \textit{ for a.a. } x \in \Omega \textit{ and all } v \in \mathbb{R}^d.$$

(b) *There exists $\mathfrak{p}^\star \in (\gamma p, p_L^*/\varpi')$ such that $\displaystyle\limsup_{|v|\to 0} \frac{|f(x,v)\cdot v|}{w(x)|v|^{\mathfrak{p}^\star}} < \infty$, uniformly a.e. in Ω.*

(c) $\displaystyle\int_\Omega F(x,u_1)dx > \frac{1}{p}\left(\frac{\mathscr{M}(\lambda_1)}{s\lambda_1^\gamma} - 1\right).$

Note that, in the more familiar and standard setting in the literature in which $L = \gamma = 1$ and $w \in L^\infty(\Omega)$, the exponent $\mathfrak{p}^\star$ in $(\mathcal{F})$–(b) belongs to the open interval (p, p^*). Furthermore, in condition $(\mathcal{F})$–(c), the constant $\displaystyle\frac{1}{p}\left(\frac{\mathscr{M}(\lambda_1)}{s\lambda_1^\gamma} - 1\right)$ is non–negative thanks to $(\mathcal{M})$. Thus, $(\mathcal{F})$–(c) is automatic when $M \equiv 1$, $s = \gamma = 1$ and $F(x,v) > 0$ a.e. in Ω for all $v \in \mathbb{R}^d \setminus \{0\}$.

An *example* of function $f : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ verifying $(\mathcal{F})$–(a) and (b) is

$$f(x,v) = w(x) \begin{cases} |v|^{\mathfrak{p}^\star - 2} v, & \text{if } |v| \leq 1, \\ |v|^{q-2} v, & \text{if } |v| > 1, \end{cases}$$

with $q \in (1, \gamma p)$, $\mathfrak{p}^\star \in (\gamma p, p_L^*/\varpi')$ and w verifying (1.4). More precisely, $(\mathcal{F})$–(a) holds with $C_f = 1$ and $(\mathcal{F})$–(b) is trivially verified. Finally, $F(x,v) > 0$ for a.a.

$x \in \Omega$ and all $v \in \mathbb{R}^d \setminus \{0\}$. As already noted, this shows that $(\mathcal{F})$–(c) holds when $M \equiv 1$ and $s = \gamma = 1$.

Following [3], in Section 2.4, we introduce in place of $(\mathcal{F})$–(c) the weaker assumption $(\mathcal{F})$–$(c)'$ much easier to check. It plays the same role for a less involved problem.

Proposition 2.2. *Assume that* $(\mathcal{F})$–(a) *and* (b) *hold. Then* $f(x,0) = 0$ *for a.a.* $x \in \Omega$*,*

$$S_f = \operatorname*{ess\,sup}_{v\neq 0,\, x\in\Omega} \frac{|f(x,v)\cdot v|}{w(x)|v|^{\gamma p}} \in \mathbb{R}^+ \quad \textit{and} \quad \operatorname*{ess\,sup}_{v\neq 0,\, x\in\Omega} \frac{|F(x,v)|}{w(x)|v|^{\gamma p}} \leq \frac{S_f}{\gamma p}. \tag{2.5}$$

Moreover, there exists $K > 0$ *such that*

$$|F(x,v)| \leq K w(x)|v|^{\mathfrak{p}^\star} \tag{2.6}$$

for a.a. $x \in \Omega$ *and all* $v \in \mathbb{R}^d$*.*

Proof. Assume first by contradiction that there exists $A \subset \Omega$, $|A| > 0$, such that $|f(x,0)| > 0$ and $w(x) > 0$ for all $x \in A$. In particular,

$$\lim_{|v|\to 0} \frac{|f(x,v)\cdot v|}{w(x)|v|^{\mathfrak{p}^\star}} = \infty$$

for all $x \in A$, contradicting $(\mathcal{F})$–(b). Hence $f(x,0) = 0$ for a.a. $x \in \Omega$.

Since $F(x,0) = 0$ a.e. in Ω by $(\mathcal{F})$, we assert that

$$\limsup_{|v|\to 0} \frac{|F(x,v)|}{w(x)|v|^{\mathfrak{p}^\star}} = \ell_0 < \infty \quad \text{uniformly a.e. in } \Omega. \tag{2.7}$$

Indeed, by $(\mathcal{F})$–(b) there exist $\ell > 0$ and $\delta > 0$ such that

$$\frac{|F(x,v)|}{w(x)|v|^{\mathfrak{p}^\star}} \leq \int_0^1 \frac{|f(x,tv)\cdot tv|}{w(x)|tv|^{\mathfrak{p}^\star}} t^{\mathfrak{p}^\star - 1} dt < \frac{\ell}{\mathfrak{p}^\star}$$

for all $v \in \mathbb{R}^d$, with $0 < |v| < \delta$, and uniformly a.e. in Ω. This implies (2.7).

Clearly, S_f defined in (2.5) is positive, being $f \not\equiv 0$. We claim that $S_f < \infty$. Indeed, uniformly a.e. in Ω

$$\lim_{|v|\to 0} \frac{|f(x,v)\cdot v|}{w(x)|v|^{\gamma p}} = \lim_{|v|\to 0} \left[\frac{|f(x,v)\cdot v|}{w(x)|v|^{\mathfrak{p}^\star}}\right] |v|^{\mathfrak{p}^\star - \gamma p} = 0$$

by $(\mathcal{F})$–(b) and the fact that $\gamma p < \mathfrak{p}^\star$. Moreover, $|f(x,v)\cdot v|/w(x)|v|^{\gamma p} \leq 2C_f|v|^{q-\gamma p}$ for a.a. $x \in \Omega$ and all $|v| \geq 1$ by $(\mathcal{F})$–(a), that is

$$\lim_{|v|\to\infty} \frac{|f(x,v)\cdot v|}{w(x)|v|^{\gamma p}} = 0 \quad \text{uniformly a.e. in } \Omega,$$

since $q < \gamma p$. This shows the claim.

Condition $(2.5)_1$ implies at once $(2.5)_2$, since

$$\frac{|F(x,v)|}{w(x)|v|^{\gamma p}} \leq \int_0^1 \frac{|f(x,tv)\cdot tv|}{w(x)|tv|^{\gamma p}} t^{\gamma p - 1} dt \leq \frac{S_f}{\gamma p}$$

for a.a. $x \in \Omega$ and all $v \in \mathbb{R}^d \setminus \{0\}$.

Finally, by (2.7) there exists $\delta > 0$ such that $|F(x,v)| \le (\ell_0 + 1)w(x)|v|^{\mathfrak{p}^\star}$ for a.a. $x \in \Omega$ and all v, with $0 < |v| < \delta$. Fix v, with $|v| \ge \delta$, then by (2.5) for a.a. $x \in \Omega$

$$|F(x,v)| \le \frac{S_f}{\gamma p}|v|^{\gamma p - \mathfrak{p}^\star} w(x)|v|^{\mathfrak{p}^\star} \le \frac{S_f \delta^{\gamma p - \mathfrak{p}^\star}}{\gamma p} w(x)|v|^{\mathfrak{p}^\star},$$

being $\mathfrak{p}^\star \in (\gamma p, p_L^*/\varpi')$ by $(\mathcal{F})$–(b). Hence, taking $K = \max\{\ell_0 + 1, S_f \delta^{\gamma p - \mathfrak{p}^\star}/\gamma p\}$, we get (2.6). □

Now we are ready to introduce *the crucial positive number*

$$\lambda_\star = \frac{s\gamma\lambda_1^\gamma}{\gamma + c_{\gamma p} S_f \lambda_1^\gamma}, \tag{2.8}$$

which is well defined by (2.4), and Propositions 2.1 and 2.2. In passing, we point out that $\lambda_\star$ coincides with the same parameter $\lambda_\star$ of [3], when $d = s = \gamma = 1$ and $M \equiv 1$, see (2.4).

2.3. *Some lemmas*

In this subsection we present some auxiliary results which are useful in applying Theorem Appendix A.1 to (1.1). In what follows, the dual space of $[W_0^{L,p}(\Omega)]^d$ is denoted by $\Big([W_0^{L,p}(\Omega)]^d\Big)^\star$.

Lemma 2.2. *The functional* $\Phi : [W_0^{L,p}(\Omega)]^d \to \mathbb{R}$, *defined by*

$$\Phi(u) = \frac{1}{p}\mathscr{M}(\|u\|^p) \tag{2.9}$$

is convex, weakly lower semi–continuous in $[W_0^{L,p}(\Omega)]^d$ *and of class* $C^1([W_0^{L,p}(\Omega)]^d)$.

Moreover, $\Phi' : [W_0^{L,p}(\Omega)]^d \to \Big([W_0^{L,p}(\Omega)]^d\Big)^\star$ *verifies the* $(\mathscr{S}_+)$ *condition, i.e. for every sequence* $(u_k)_k \subset [W_0^{L,p}(\Omega)]^d$ *such that* $u_k \rightharpoonup u$ *in* $[W_0^{L,p}(\Omega)]^d$ *and*

$$\limsup_{k\to\infty} M(\|u_k\|^p) \int_\Omega |\mathcal{D}_L u_k|^{p-2}\mathcal{D}_L u_k \cdot (\mathcal{D}_L u_k - \mathcal{D}_L u)dx \le 0, \tag{2.10}$$

then $u_k \to u$ *in* $[W_0^{L,p}(\Omega)]^d$.

Proof. A simple calculation shows that Φ is convex in $[W_0^{L,p}(\Omega)]^d$, since $\mathscr{M}$ is convex and monotone non–decreasing in $\mathbb{R}_0^+$ by $(\mathcal{M})$. Moreover, we claim that $\Phi \in C^1([W_0^{L,p}(\Omega)]^d)$. Indeed, Φ is Gâteaux differentiable in $[W_0^{L,p}(\Omega)]^d$ and for all $u, v \in [W_0^{L,p}(\Omega)]^d$

$$\langle \Phi'(u), v\rangle = M(\|u\|^p) \int_\Omega |\mathcal{D}_L u|^{p-2}\mathcal{D}_L u \cdot \mathcal{D}_L v\, dx.$$

Now, let u, $(u_k)_k \subset [W_0^{L,p}(\Omega)]^d$ be such that $u_k \to u$ in $[W_0^{L,p}(\Omega)]^d$ as $k \to \infty$. We claim that

$$\|\Phi'(u_k) - \Phi'(u)\|_\star = \sup_{\substack{v \in [W_0^{L,p}(\Omega)]^d \\ \|v\|=1}} |\langle \Phi'(u_k) - \Phi'(u), v\rangle| = o(1)$$

as $k \to \infty$. By Hölder's inequality

$$\begin{aligned} &\big|\langle \Phi'(u_k) - \Phi'(u), v\rangle\big| \\ &\qquad \le \big\|M(\|u_k\|^p)|\mathcal{D}_L u_k|^{p-2}\mathcal{D}_L u_k - M(\|u\|^p)|\mathcal{D}_L u|^{p-2}\mathcal{D}_L u\big\|_{N,p'} \|\mathcal{D}_L v\|_{N,p}. \end{aligned}$$

Hence

$$\begin{aligned} &\|\Phi'(u_k) - \Phi'(u)\|_\star \\ &\qquad \le \big\|M(\|u_k\|^p)|\mathcal{D}_L u_k|^{p-2}\mathcal{D}_L u_k - M(\|u\|^p)|\mathcal{D}_L u|^{p-2}\mathcal{D}_L u\big\|_{N,p'}. \end{aligned} \tag{2.11}$$

Fix now a subsequence $(u_{k_j})_j$ of $(u_k)_k$. Clearly, $u_{k_j} \to u$ in $[W_0^{L,p}(\Omega)]^d$ and so $\mathcal{D}_L u_{k_j} \to \mathcal{D}_L u$ in $[L^p(\Omega)]^N$ as $j \to \infty$, where as usual $N = d$ if L is even and $N = dn$ if L is odd. By Lemma Appendix A.1, with $m = N$, $\sigma = p$ and $\omega \equiv 1$, there exist a subsequence of $(u_{k_j})_j$, still denoted by $(u_{k_j})_j$, and an appropriate function $h \in L^p(\Omega)$ such that a.e. in Ω we get that $\mathcal{D}_L u_{k_j} \to \mathcal{D}_L u$ as $j \to \infty$ and $|\mathcal{D}_L u_{k_j}| \le h$ for all $j \in \mathbb{N}$. Thus,

$$\begin{aligned} &\big|M(\|u_{k_j}\|^p)|\mathcal{D}_L u_{k_j}|^{p-2}\mathcal{D}_L u_{k_j} - M(\|u\|^p)|\mathcal{D}_L u|^{p-2}\mathcal{D}_L u\big|^{p'} \\ &\qquad\qquad \le 2^{p'-1}\left\{\left[M(\|u_{k_j}\|^p)|\mathcal{D}_L u_{k_j}|^{p-1}\right]^{p'} + \left[M(\|u\|^p)|\mathcal{D}_L u|^{p-1}\right]^{p'}\right\} \\ &\qquad\qquad \le (2K)^{p'} h^p \in L^1(\Omega), \end{aligned}$$

where $K = \sup_j M(\|u_{k_j}\|^p) < \infty$, being $(u_k)_k$ convergent and so bounded in $[W_0^{L,p}(\Omega)]^d$. In particular, $M(\|u_{k_j}\|^p) \to M(\|u\|^p)$ by $(\mathcal{M})$. Furthermore, $|\mathcal{D}_L u_{k_j}|^{p-2}\mathcal{D}_L u_{k_j} \to |\mathcal{D}_L u|^{p-2}\mathcal{D}_L u$ a.e. in Ω. Therefore,

$$\begin{aligned} &\big|M(\|u_{k_j}\|^p)|\mathcal{D}_L u_{k_j}|^{p-2}\mathcal{D}_L u_{k_j} - M(\|u\|^p)|\mathcal{D}_L u|^{p-2}\mathcal{D}_L u\big| \\ &\quad \le K\big||\mathcal{D}_L u_{k_j}|^{p-2}\mathcal{D}_L u_{k_j} - |\mathcal{D}_L u|^{p-2}\mathcal{D}_L u\big| + |M(\|u_{k_j}\|^p) - M(\|u\|^p)| \cdot |\mathcal{D}_L u|^{p-1} \\ &\quad \to 0 \quad \text{a.e. in } \Omega, \text{ as } j \to \infty. \end{aligned}$$

Applying the Lebesgue dominated convergence theorem, we obtain as $j \to \infty$

$$\|M(\|u_{k_j}\|^p)|\mathcal{D}_L u_{k_j}|^{p-2}\mathcal{D}_L u_{k_j} - M(\|u\|^p)|\mathcal{D}_L u|^{p-2}\mathcal{D}_L u\|_{N,p'} \to 0. \tag{2.12}$$

Actually, (2.12) holds for the entire sequence $(u_k)_k$ and this implies the claim, by virtue of (2.11).

Therefore Φ is of class $C^1([W_0^{L,p}(\Omega)]^d)$. In particular, Φ is weakly lower semi–continuous in $[W_0^{L,p}(\Omega)]^d$, by Corollary 3.9 of [30].

Let us now prove the $(\mathscr{S}_+)$ condition. Let $(u_k)_k \subset [W_0^{L,p}(\Omega)]^d$ be such that $u_k \rightharpoonup u$ in $[W_0^{L,p}(\Omega)]^d$ and (2.10) holds. Since $u_k \rightharpoonup u$, then

$$\lim_{k\to\infty} M(\|u\|^p)\int_\Omega |\mathcal{D}_L u|^{p-2}\mathcal{D}_L u \cdot \mathcal{D}_L(u_k - u)dx = 0, \tag{2.13}$$

being $|\mathcal{D}_L u|^{p-2}\mathcal{D}_L u \in [L^{p'}(\Omega)]^N$. Hence (2.10) is equivalent to

$$\limsup_{k\to\infty}\int_\Omega \big[M(\|u_k\|^p)|\mathcal{D}_L u_k|^{p-2}\mathcal{D}_L u_k - M(\|u\|^p)|\mathcal{D}_L u|^{p-2}\mathcal{D}_L u\big]\cdot \mathcal{D}_L(u_k-u)dx \le 0.$$

Since $\mathscr{M}(\|\cdot\|^p)$ is convex in $[W_0^{L,p}(\Omega)]^d$, then

$$\big[M(\|u_k\|^p)|\mathcal{D}_L u_k|^{p-2}\mathcal{D}_L u_k - M(\|u\|^p)|\mathcal{D}_L u|^{p-2}\mathcal{D}_L u\big]\cdot \mathcal{D}_L(u_k-u) \ge 0.$$

Therefore

$$\lim_{k\to\infty}\int_\Omega \big[M(\|u_k\|^p)|\mathcal{D}_L u_k|^{p-2}\mathcal{D}_L u_k - M(\|u\|^p)|\mathcal{D}_L u|^{p-2}\mathcal{D}_L u\big]\cdot \mathcal{D}_L(u_k-u)dx = 0.$$

This implies by (2.13)

$$\lim_{k\to\infty} M(\|u_k\|^p)\int_\Omega |\mathcal{D}_L u_k|^{p-2}\mathcal{D}_L u_k\cdot\mathcal{D}_L(u_k-u)\,dx = 0. \tag{2.14}$$

Now, two cases arise.

<u>*Case* $u \ne 0$</u>. By the weak lower semi–continuity of the norm, we get

$$0 < \|u\| \le \liminf_k \|u_k\| = \ell$$

and consequently there exists $K \in \mathbb{N}$ such that $\|u_k\| \ge \ell/2 > 0$ for all $k \ge K$. Hence, by condition $(\mathcal{M})$

$$M(\|u_k\|^p) \ge \kappa > 0 \quad \text{for all } k \ge K, \tag{2.15}$$

with $\kappa = s\gamma(\ell/2)^{p(\gamma-1)}$. Thus, by (2.14) and (2.15), we have

$$\lim_{k\to\infty}\int_\Omega |\mathcal{D}_L u_k|^{p-2}\mathcal{D}_L u_k\cdot\mathcal{D}_L(u_k-u)dx = 0. \tag{2.16}$$

By convexity

$$\|\mathcal{D}_L u\|_{N,p}^p + p\int_\Omega |\mathcal{D}_L u_k|^{p-2}\mathcal{D}_L u_k\cdot\mathcal{D}_L(u_k-u)dx \ge \|\mathcal{D}_L u_k\|_{N,p}^p. \tag{2.17}$$

Therefore, combining together (2.16), (2.17) and the weak lower semi–continuity of the norm, we have

$$\|\mathcal{D}_L u\|_{N,p}^p \ge \limsup_{k\to\infty}\|\mathcal{D}_L u_k\|_{N,p}^p \ge \liminf_{k\to\infty}\|\mathcal{D}_L u_k\|_{N,p}^p \ge \|\mathcal{D}_L u\|_{N,p}^p.$$

In other words,

$$\lim_{k\to\infty}\|u_k\| = \|u\|. \tag{2.18}$$

Since $[W_0^{L,p}(\Omega)]^d$ is uniformly convex by Proposition Appendix A.2, we immediately get from (2.18) and the weak convergence $u_k \rightharpoonup u$, that

$$\lim_{k\to\infty}\|u_k - u\| = 0,$$

as required.

<u>*Case $u = 0$*</u>. Suppose first by contradiction that

$$0 = \|u\| < \liminf_{k\to\infty} \|u_k\|. \tag{2.19}$$

As before, (2.15) holds. Hence, proceeding as in the previous case, we conclude that $\lim_k u_k = 0$, which contradicts (2.19). Therefore, the only possible case is

$$0 = \|u\| = \liminf_{k} \|u_k\|.$$

Assume now by contradiction that

$$\mathcal{L} = \limsup_{k\to\infty} \|u_k\| > \liminf_{k\to\infty} \|u_k\| = \|u\| = 0. \tag{2.20}$$

In particular, there exist a subsequence $(u_{k_j})_j$ of $(u_k)_k$ and $K \in \mathbb{N}$ such that $\mathcal{L} = \lim_{j\to\infty} \|u_{k_j}\|$ and

$$M(\|u_{k_j}\|^p) \geq \kappa > 0 \quad \text{for all } j \geq K,$$

where $\kappa = s\gamma(\mathcal{L}/2)^{p(\gamma-1)}$. Consequently, the argument from (2.15) to (2.18) along the subsequence $(u_{k_j})_j$ implies that $\lim_{j\to\infty} \|u_{k_j}\| = 0$, which contradicts (2.20).

We have shown also in this case that $\limsup_{k\to\infty} \|u_k\| = \liminf_{k\to\infty} \|u_k\| = \|u\| = 0$. In other words $u_k \to 0$ in $[W_0^{L,p}(\Omega)]^d$, as required. □

Without further mentioning, we assume that $(\mathcal{F})$–(a) and $(\mathcal{F})$–(b) hold. The main result of the section is proved by using the energy functional J_λ associated to (1.1), which is given by $J_\lambda(u) = \Phi(u) + \lambda\Psi(u)$, where

$$\begin{gathered}\Psi(u) = \Psi_1(u) + \Psi_2(u), \\ \Psi_1(u) = -\frac{1}{p}\|u\|_{d,p,w}^{\gamma p}, \quad \Psi_2(u) = -\int_\Omega F(x, u(x))dx.\end{gathered} \tag{2.21}$$

Clearly, the functional J_λ is well defined in $[W_0^{L,p}(\Omega)]^d$ and of class $C^1([W_0^{L,p}(\Omega)]^d)$, see the proof of Lemma 2.2. Furthermore, for all u, $\varphi \in [W_0^{L,p}(\Omega)]^d$

$$\begin{aligned}\langle J_\lambda'(u), \varphi\rangle =& M(\|u\|^p) \int_\Omega |\mathcal{D}_L u(x)|^{p-2} \mathcal{D}_L u(x) \cdot \mathcal{D}_L \varphi(x)\, dx \\ & - \lambda \int_\Omega \Big[\gamma \|u\|_{d,p,w}^{p(\gamma-1)} w(x)|u(x)|^{p-2}u(x) + f(x, u(x))\Big] \cdot \varphi(x)\, dx.\end{aligned}$$

Therefore, the critical points $u \in [W_0^{L,p}(\Omega)]^d$ of J_λ are exactly the weak solutions of (1.1).

Using the notation of Appendix A, if $\Psi(v) < 0$ at some $v \in [W_0^{L,p}(\Omega)]^d$, then *the crucial positive number*

$$\lambda^\star = \varphi_1(0) = \inf_{u\in\Psi^{-1}(I_0)} -\frac{\Phi(u)}{\Psi(u)}, \qquad I_0 = (-\infty, 0), \tag{2.22}$$

is well defined.

Lemma 2.3. *If f satisfies also $(\mathcal{F})$–(c), then $\Psi^{-1}(I_0)$ is non–empty and moreover $\lambda_\star \leq \lambda^\star < s\lambda_1^\gamma$.*

Proof. From $(\mathcal{F})$–(c) it follows that

$$\Psi(u_1) < -\frac{\mathscr{M}(\lambda_1)}{ps\lambda_1^\gamma} < 0, \quad \text{i.e. } u_1 \in \Psi^{-1}(I_0). \tag{2.23}$$

Hence, (2.22) is meaningful. By (2.23) and Proposition 2.1

$$\lambda^\star = \inf_{u\in\Psi^{-1}(I_0)} -\frac{\Phi(u)}{\Psi(u)} \le \frac{\Phi(u_1)}{-\Psi(u_1)} < \frac{\mathscr{M}(\|u_1\|^p)/p}{\mathscr{M}(\lambda_1)/ps\lambda_1^\gamma} = s\lambda_1^\gamma,$$

as required. Finally, by $(\mathcal{M})$, (2.3), (2.4), (2.5) and (2.21), for all $u \in \Psi^{-1}(I_0)$, we have

$$\begin{aligned}\frac{\Phi(u)}{|\Psi(u)|} &\ge \frac{\mathscr{M}(\|u\|^p)/p}{\dfrac{1}{p}\|u\|_{d,p,w}^{\gamma p} + \dfrac{S_f}{\gamma p}\|u\|_{d,\gamma p,w}^{\gamma p}} \ge \frac{s\|u\|^{\gamma p}/p}{\dfrac{1}{p\lambda_1^\gamma}\|u\|^{\gamma p} + \dfrac{c_{\gamma p}S_f}{\gamma p}\|u\|^{\gamma p}} \\ &= \frac{s\gamma\lambda_1^\gamma}{\gamma + c_{\gamma p}S_f\lambda_1^\gamma} = \lambda_\star.\end{aligned}$$

Hence, in particular $\lambda^\star \ge \lambda_\star$. □

Lemma 2.4. *The operators Ψ_1', Ψ_2', $\Psi' : [W_0^{L,p}(\Omega)]^d \to \left([W_0^{L,p}(\Omega)]^d\right)^\star$ are compact and Ψ_1, Ψ_2, Ψ are sequentially weakly continuous in $[W_0^{L,p}(\Omega)]^d$.*

Proof. Of course, $\Psi' = \Psi_1' + \Psi_2'$, where

$$\langle\Psi_1'(u), v\rangle = -\gamma\|u\|_{d,p,w}^{p(\gamma-1)}\int_\Omega w(x)|u|^{p-2}u\cdot v dx \quad \text{and} \quad \langle\Psi_2'(u), v\rangle = -\int_\Omega f(x,u)\cdot v dx$$

for all $u, v \in [W_0^{L,p}(\Omega)]^d$. Since Ψ_1' and Ψ_2' are continuous, thanks to the reflexivity of $[W_0^{L,p}(\Omega)]^d$ it is enough to show that Ψ_1' and Ψ_2' are weak–to–strong sequentially continuous, i.e. if $(u_k)_k$, u are in $[W_0^{L,p}(\Omega)]^d$ and $u_k \rightharpoonup u$ in $[W_0^{L,p}(\Omega)]^d$, then $\|\Psi_1'(u_k) - \Psi_1'(u)\|_\star \to 0$ and $\|\Psi_2'(u_k) - \Psi_2'(u)\|_\star \to 0$ as $k \to \infty$. To this aim, fix $(u_k)_k \subset [W_0^{L,p}(\Omega)]^d$, with $u_k \rightharpoonup u$ in $[W_0^{L,p}(\Omega)]^d$.

First, $u_k \to u$ in $[L^p(\Omega, w)]^d$ by Lemma 2.1–(i). Therefore, $\mathcal{N}_p(u_k) \to \mathcal{N}_p(u)$ in $[L^{p'}(\Omega, w)]^d$ by Lemma Appendix A.2.

For all $v \in [W_0^{L,p}(\Omega)]^d$, with $\|v\| = 1$, by Hölder's inequality and (2.3),

$$\begin{aligned}|\langle\Psi_1'(u_k) - \Psi_1'(u), v\rangle| &\le \gamma\|u_k\|_{d,p,w}^{p(\gamma-1)}\int_\Omega w(x)^{1/p'}|\mathcal{N}_p(u_k) - \mathcal{N}_p(u)|w(x)^{1/p}|v|dx \\ &\qquad + \gamma\left|\|u_k\|_{d,p,w}^{p(\gamma-1)} - \|u\|_{d,p,w}^{p(\gamma-1)}\right|\|u\|_{d,p,w}^{p-1}\|v\|_{d,p,w} \\ &\le \gamma\mathcal{C}\left\{\|\mathcal{N}_p(u_k) - \mathcal{N}_p(u)\|_{d,p',w} + \left|\|u_k\|_{d,p,w}^{p(\gamma-1)} - \|u\|_{d,p,w}^{p(\gamma-1)}\right|\right\}\|v\|_{d,p,w} \\ &\le \lambda_1^{-1/p}\gamma\mathcal{C}\left\{\|\mathcal{N}_p(u_k) - \mathcal{N}_p(u)\|_{d,p',w} + \left|\|u_k\|_{d,p,w}^{p(\gamma-1)} - \|u\|_{d,p,w}^{p(\gamma-1)}\right|\right\},\end{aligned}$$

where $\mathcal{C} = \sup_k \|u_k\|_{d,p,w}^{p(\gamma-1)}$. Hence, $\|\Psi_1'(u_k) - \Psi_1'(u)\|_\star \to 0$ as $k \to \infty$ and Ψ_1' is compact.

Similarly, $u_k \to u$ in $[L^q(\Omega, w)]^d$ by Lemma 2.1–(i). Thus, $\mathcal{N}_f(u_k) \to \mathcal{N}_f(u)$ as $k \to \infty$ in $[L^{q'}(\Omega, w^{1/(1-q)})]^d$ by Lemma Appendix A.2.

Finally, for all $v \in [W_0^{L,p}(\Omega)]^d$, with $\|v\| = 1$, we have again

$$
\begin{aligned}
|\langle \Psi_2'(u_k) - \Psi_2'(u), v\rangle| &\le \int_\Omega w(x)^{-1/q} |\mathcal{N}_f(u_k) - \mathcal{N}_f(u)| w^{1/q} |v| dx \\
&\le \|\mathcal{N}_f(u_k) - \mathcal{N}_f(u)\|_{d,q',w^{1/(1-q)}} \|v\|_{d,q,w} \\
&\le \mathcal{S}_{d,q,w} \|\mathcal{N}_f(u_k) - \mathcal{N}_f(u)\|_{d,q',w^{1/(1-q)}}.
\end{aligned}
$$

Thus, $\|\Psi_2'(u_k) - \Psi_2'(u)\|_\star \to 0$ as $k \to \infty$, that is Ψ_2' is compact.

Consequently, $\Psi' = \Psi_1' + \Psi_2'$ is compact, then Ψ is sequentially weakly continuous by Corollary 41.9 of [31], being $[W_0^{L,p}(\Omega)]^d$ reflexive. □

Lemma 2.5. *The functional* $J_\lambda(u) = \Phi(u) + \lambda\Psi(u)$ *is coercive for all* λ *in the interval* $(-\infty, s\lambda_1^\gamma)$.

Proof. Fix $\lambda \in (-\infty, s\lambda_1^\gamma)$. Then by $(\mathcal{M})$, (2.3) and $(\mathcal{F})$–(a) for all $u \in [W_0^{L,p}(\Omega)]^d$, with $\|u\| \ge 1$,

$$
\begin{aligned}
J_\lambda(u) &\ge \frac{1}{p}\mathscr{M}(\|u\|^p) - \frac{\lambda}{p}\|u\|_{d,p,w}^{\gamma p} - |\lambda| \int_\Omega |F(x,u)| dx \\
&\ge \frac{1}{p}\left(\mathscr{M}(\|u\|^p) - \frac{\lambda^+}{\lambda_1^\gamma}\|u\|^{\gamma p}\right) - |\lambda| C_f \int_\Omega \left(w(x)|u| + \frac{w(x)}{q}|u|^q\right) dx \\
&\ge \frac{1}{p}\left(s - \frac{\lambda^+}{\lambda_1^\gamma}\right)\|u\|^{\gamma p} - |\lambda| C_f \left\{\int_{\Omega_1} w(x) dx + \int_{\Omega_2} w(x)|u|^q dx + \frac{1}{q}\int_\Omega w(x)|u|^q dx\right\} \\
&\ge \frac{1}{p}\left(s - \frac{\lambda^+}{\lambda_1^\gamma}\right)\|u\|^{\gamma p} - |\lambda| C_1 - |\lambda|\, C_2 \|u\|^q,
\end{aligned}
$$

where $\Omega_1 = \{x \in \Omega : |u(x)| \le 1\}$, $\Omega_2 = \{x \in \Omega : |u(x)| \ge 1\}$, $C_1 = C_f\|w\|_1$ and $C_2 = C_f \mathcal{S}_{d,q,w}^q (q+1)/q$. This completes the proof, since $1 < q < \gamma p$ by $(\mathcal{F})$–(a). □

2.4. *The existence and multiplicity results for* (1.1)

Thanks to the results of Section 2.3 all the structural assumptions $(\mathcal{H}_1)$–$(\mathcal{H}_4)$ of Theorem Appendix A.1 are clearly verified by J_λ. Thus we are now able to prove

Theorem 2.1. *Let* $(\mathcal{F})$–(a), (b) *hold, and let* $\lambda_\star$ *be the number defined in* (2.8), *while* $\lambda^\star = \varphi_1(0) < s\lambda_1^\gamma$ *is given in* (2.22).

(i) *If* $\lambda \in [0, \lambda_\star)$, *then* (1.1) *has only the trivial solution.*

(ii) *If furthermore* $(\mathcal{F})$–(c) *holds and* $q \in (1, p)$ *in* $(\mathcal{F})$–(a), *then* (1.1) *admits at least two nontrivial solutions for every* $\lambda \in (\lambda^\star, s\lambda_1^\gamma)$.

Proof. (i) Let $u \in [W_0^{L,p}(\Omega)]^d$ be a nontrivial weak solution of the problem (1.1), then

$$
\begin{aligned}
s\gamma\lambda_1^\gamma\|u\|^{\gamma p} &\le \lambda_1^\gamma M(\|u\|^p)\|u\|^p = \lambda_1^\gamma\lambda\int_\Omega\{\gamma\|u\|_{d,p,w}^{p(\gamma-1)}w(x)|u|^p + f(x,u)\cdot u\}dx \\
&\le \lambda_1^\gamma\lambda\left(\gamma\|u\|_{d,p,w}^{\gamma p} + \int_\Omega \frac{|f(x,u)\cdot u|}{w(x)|u|^{\gamma p}}w(x)|u|^{\gamma p}dx\right) \\
&\le \lambda_1^\gamma\lambda(\gamma\|u\|_{d,p,w}^{\gamma p} + S_f\|u\|_{d,\gamma p,w}^{\gamma p}) \\
&\le \lambda\left(\gamma + c_{\gamma p}S_f\lambda_1^\gamma\right)\|u\|^{\gamma p}
\end{aligned}
$$

by $(\mathcal{M})$, (2.3), (2.4) and $(2.5)_1$. Therefore $\lambda \ge \lambda_\star$, as required.

(ii) By Lemmas 2.2–2.5 the functional J_λ verifies all the structural assumptions $(\mathcal{H}_1)$–$(\mathcal{H}_4)$ of Theorem Appendix A.1, with $I = (-\infty, s\lambda_1^\gamma)$. It remains to show (A.2). We claim that $\Psi([W_0^{L,p}(\Omega)]^d) \supset \mathbb{R}_0^-$. Indeed, $\Psi(0) = 0$ and arguing as in the proof of Lemma 2.5, we get the following estimate

$$
\left|\int_\Omega F(x,u(x))dx\right| \le c(\|w\|_1 + \|u\|_{d,q,w}^q),
$$

where $c = C_f(1+q)/q$. Furthermore, by Hölder's inequality

$$
\|u\|_{d,q,w}^q \le \|w\|_1^{(p-q)/p}\|u\|_{d,p,w}^q,
$$

since $1 < q < p$ and $w \in L^1(\Omega)$, being $\varpi > 1$ and Ω bounded. Hence, combining together the previous inequalities, we get

$$
\Psi(u) \le -\frac{1}{p}\|u\|_{d,p,w}^{\gamma p} + c\|w\|_1 + c\|w\|_1^{(p-q)/p}\|u\|_{d,p,w}^q.
$$

Therefore,

$$
\lim_{\substack{u\in[W_0^{L,p}(\Omega)]^d \\ \|u\|_{d,p,w}\to\infty}} \Psi(u) = -\infty,
$$

being $q < p \le \gamma p$. Hence, the claim follows by the continuity of Ψ.

In particular, $(\inf\Psi, \sup\Psi) \supset \mathbb{R}_0^-$. Now, for every $u \in \Psi^{-1}(I_0)$ we have

$$
\varphi_1(r) \le \frac{\displaystyle\inf_{v\in\Psi^{-1}(r)}\Phi(v) - \Phi(u)}{\Psi(u) - r} \le -\frac{\Phi(u)}{\Psi(u) - r}
$$

for all $r \in (\Psi(u), 0)$, so that

$$
\limsup_{r\to 0^-}\varphi_1(r) \le -\frac{\Phi(u)}{\Psi(u)} \quad \text{for all } u \in \Psi^{-1}(I_0),
$$

in other words,

$$
\limsup_{r\to 0^-}\varphi_1(r) \le \varphi_1(0) = \lambda^\star. \tag{2.24}
$$

Now, by Lemma 2.1–(i), the fact that $\mathfrak{p}^\star < p_L^*/\varpi'$ and (2.6),

$$
|\Psi(u)| \le \frac{1}{p}\|u\|_{d,p,w}^{\gamma p} + K\|u\|_{d,\mathfrak{p}^\star,w}^{\mathfrak{p}^\star} \le \frac{1}{p}\|u\|_{d,p,w}^{\gamma p} + \mathfrak{K}\|u\|^{\mathfrak{p}^\star} \tag{2.25}
$$

for every $u \in [W_0^{L,p}(\Omega)]^d$, where $\mathfrak{K} = K\mathcal{S}_{d,\mathfrak{p}^\star,w}^{\mathfrak{p}^\star} > 0$. Therefore, for $r < 0$ and $v \in \Psi^{-1}(r)$,

$$r = \Psi(v) \geq -\frac{1}{p\lambda_1^\gamma}\|v\|^{\gamma p} - \mathfrak{K}\|v\|^{\mathfrak{p}^\star} \geq -\frac{1}{s\lambda_1^\gamma}\Phi(v) - \mathfrak{K}\left(\frac{p}{s}\right)^{\mathfrak{p}^\star/\gamma p}\Phi(v)^{\mathfrak{p}^\star/\gamma p} \quad (2.26)$$

by $(\mathcal{M})$, (2.3), (2.9) and (2.25). By Lemma 2.2 and Proposition Appendix A.2 the functional Φ is bounded below, coercive and lower semi–continuous in the reflexive Banach space $[W_0^{L,p}(\Omega)]^d$. Hence, it is easy to see that Φ is also coercive in the sequentially weakly closed non–empty set $\Psi^{-1}(r)$. Therefore, by Theorem 6.1.1 of [29], there exists an element

$$u_r \in \Psi^{-1}(r) \quad \text{such that } \Phi(u_r) = \inf_{v\in\Psi^{-1}(r)}\Phi(v).$$

By (A.1) we have

$$\varphi_2(r) \geq -\frac{1}{r}\inf_{v\in\Psi^{-1}(r)}\Phi(v) = \frac{\Phi(u_r)}{|r|},$$

being $0 \in \Psi^{-1}(I^r)$. From (2.26) we get

$$\begin{aligned}1 &\leq \frac{1}{s\lambda_1^\gamma}\cdot\frac{\Phi(u_r)}{|r|} + \mathfrak{K}\left(\frac{p}{s}\right)^{\mathfrak{p}^\star/\gamma p}|r|^{\mathfrak{p}^\star/\gamma p-1}\left(\frac{\Phi(u_r)}{|r|}\right)^{\mathfrak{p}^\star/\gamma p}\\ &\leq \frac{\varphi_2(r)}{s\lambda_1^\gamma} + \mathfrak{K}\left(\frac{p}{s}\right)^{\mathfrak{p}^\star/\gamma p}|r|^{\mathfrak{p}^\star/\gamma p-1}\varphi_2(r)^{\mathfrak{p}^\star/\gamma p}.\end{aligned}$$

There are now two possibilities to be considered: either φ_2 is locally bounded at 0^-, so that the above inequality shows at once that

$$\liminf_{r\to 0^-}\varphi_2(r) \geq s\lambda_1^\gamma,$$

being $\mathfrak{p}^\star > \gamma p$ by $(\mathcal{F})$–(b), or $\limsup_{r\to 0^-}\varphi_2(r) = \infty$. In both cases (2.24) and Lemma 2.3 yield that for all integers $k \geq k^\star = 1 + [2/(s\lambda_1^\gamma - \lambda^\star)]$ there exists a number $r_k < 0$ so close to zero that $\varphi_1(r_k) < \lambda^\star + 1/k < s\lambda_1^\gamma - 1/k < \varphi_2(r_k)$, that is (A.2) holds. Hence, by Theorem Appendix A.1–(ii), Part (a), being $u \equiv 0$ a critical point of J_λ, problem (1.1) admits at least two nontrivial solutions for all

$$\lambda \in \bigcup_{k=k^\star}^{\infty}(\varphi_1(r_k),\varphi_2(r_k)) \cap I \supset \bigcup_{k=k^\star}^{\infty}[\lambda^\star + 1/k, s\lambda_1^\gamma - 1/k] = (\lambda^\star, s\lambda_1^\gamma),$$

as claimed. □

We now consider the simpler problem

$$\begin{cases} M(\|u\|^p)\Delta_p^L u = \lambda f(x,u) & \text{in } \Omega,\\ D^\alpha u_k\big|_{\partial\Omega} = 0 \quad \text{for all } \alpha, \text{ with } |\alpha| \leq L-1, \text{ and for all } k = 1,\dots,d, \end{cases} \quad (2.27)$$

where f verifies condition $(\mathcal{F})$, with $(\mathcal{F})$–(c) replaced by the less stringent assumption

$(\mathcal{F})$–$(c)'$ *Assume that there exist* $x_0 \in \Omega$, $v_0 \in \mathbb{R}^d$ *and* $r_0 > 0$ *so small that the closed ball* $B_0 = \{x \in \mathbb{R}^n : |x - x_0| \le r_0\}$ *is contained in* Ω *and*

$$\operatorname*{ess\,inf}_{B_0} F(x, v_0) = \mu_0 > 0, \qquad \operatorname*{ess\,sup}_{B_0} \max_{|v| \le |v_0|} |F(x, v)| = M_0 < \infty.$$

Clearly, when f does not depend on x, condition $(\mathcal{F})$–$(c)'$ simply reduces to the request that $F(v_0) > 0$ at a point $v_0 \in \mathbb{R}^d$. This case is interesting also because the unpleasant restriction $q \in (1, p)$ requested in Theorem 2.1–(ii) can be avoided. In this new setting, the next theorem extends the main result of [17] to the p–Laplace operator also for $p \in (1, 2)$, and Corollary 3.6 of [3] to higher order operators, involving the Kirchhoff function.

Theorem 2.2. *Let* $(\mathcal{F})$–(a), (b) *hold, and let* $\ell_\star = s\gamma / c_{\gamma p} S_f$.

(i) *If* $\lambda \in [0, \ell_\star)$, *then* (2.27) *has only the trivial solution.*
(ii) *If furthermore* $(\mathcal{F})$–$(c)'$ *holds, then there exists* $\ell^\star \ge \ell_\star$ *such that* (2.27) *admits at least two nontrivial solutions for all* $\lambda \in (\ell^\star, \infty)$.

Proof. The part (i) of the statement is proved by using the same argument produced for the proof of Theorem 2.1–(i), being

$$s\gamma \|u\|^{\gamma p} \le M(\|u\|^p)\|u\|^p = \lambda \int_\Omega f(x, u) \cdot u \, dx \le \lambda S_f \|u\|^{\gamma p}_{d,\gamma p, w} \le \lambda c_{\gamma p} S_f \|u\|^{\gamma p}.$$

Thus, if u is a nontrivial weak solution of (2.27), then necessarily $\lambda \ge \ell_\star$, as required.

In order to prove (ii), we consider the energy functional J_λ associated to (2.27), given by $J_\lambda(u) = \Phi(u) + \lambda \Psi_2(u)$, where Φ is defined in the statement of Lemma 2.2 and Ψ_2 in (2.21). We claim that J_λ is coercive for every $\lambda \in \mathbb{R}$. Indeed, as shown in the proof of Lemma 2.5, for all $u \in [W_0^{L,p}(\Omega)]^d$, with $\|u\| \ge 1$,

$$J_\lambda(u) \ge \frac{s}{p}\|u\|^{\gamma p} - |\lambda| C_1 - |\lambda| \, C_2 \|u\|^q,$$

where C_1, C_2 are the constants determined in the proof of Lemma 2.5. This shows the claim, since $1 < q < \gamma p$ by $(\mathcal{F})$–(a). Hence, here $I = \mathbb{R}$.

Next, we show that there exists $u_0 \in [W_0^{L,p}(\Omega)]^d$ such that $\Psi_2(u_0) < 0$. Note that $v_0 \ne 0$ in $(\mathcal{F})$–$(c)'$. Take $\sigma \in (0, 1)$ and put $B = \{x \in \mathbb{R}^n : |x - x_0| \le \sigma r_0\}$. Of course, $B \subset B_0$. Consider a function $u_0 \in [C_0^\infty(\Omega)]^d$ such that

$$|u_0| \le |v_0| \text{ in } \Omega, \quad \operatorname{supp} u_0 \subset B_0 \quad \text{and} \quad u_0 \equiv v_0 \text{ in } B.$$

Clearly, $u_0 \in [W_0^{L,p}(\Omega)]^d$. Now, by $(\mathcal{F})$–$(c)'$,

$$\begin{aligned}\Psi_2(u_0) &= -\int_{B_0 \setminus B} F(x, u_0(x)) dx - \int_B F(x, v_0) dx \le M_0 |B_0 \setminus B| - \mu_0 |B| \\ &= \omega_n r_0^n \left[M_0(1 - \sigma^n) - \mu_0 \sigma^n\right],\end{aligned}$$

where ω_n is the measure of the unit ball in $\mathbb{R}^n$. Therefore, taking $\sigma \in (0,1)$ so large that $\sigma^n > M_0/(\mu_0 + M_0)$, we get that $\Psi_2(u_0) < 0$, as claimed. Hence, the crucial number

$$\ell^\star = \varphi_1(0) = \inf_{u\in\Psi_2^{-1}(I_0)} -\frac{\Phi(u)}{\Psi_2(u)}, \quad I_0 = (-\infty, 0), \tag{2.28}$$

is well defined.

Furthermore, as in the proof of Lemma 2.3, for all $u \in \Psi_2^{-1}(I_0)$, we have

$$\frac{\Phi(u)}{|\Psi_2(u)|} \geq \frac{\mathscr{M}(\|u\|^p)/p}{c_{\gamma p} S_f \|u\|^{\gamma p}/\gamma p} \geq \frac{s\gamma\|u\|^{\gamma p}}{c_{\gamma p} S_f \|u\|^{\gamma p}} = \frac{s\gamma}{c_{\gamma p} S_f} = \ell_\star.$$

Hence, $\ell^\star \geq \ell_\star$ by (2.28).

In particular, for all $u \in \Psi_2^{-1}(I_0)$, it results

$$\varphi_1(r) \leq \frac{\inf\limits_{v\in\Psi_2^{-1}(r)} \Phi(v) - \Phi(u)}{\Psi_2(u) - r} \leq -\frac{\Phi(u)}{\Psi_2(u) - r}$$

for all $r \in (\Psi_2(u), 0)$. Thus,

$$\limsup_{r\to 0^-} \varphi_1(r) \leq \varphi_1(0) = \ell^\star. \tag{2.29}$$

Here (2.25) simply reduces to

$$|\Psi_2(u)| \leq \mathfrak{K}\|u\|^{\mathfrak{p}^\star}.$$

Taken $r < 0$ and $v \in \Psi_2^{-1}(r)$, we obtain by (2.9)

$$|r| = |\Psi_2(v)| \leq \mathfrak{K}\|v\|^{\mathfrak{p}^\star} \leq \mathfrak{K}\left(\frac{p}{s}\right)^{\mathfrak{p}^\star/\gamma p} \Phi(v)^{\mathfrak{p}^\star/\gamma p}.$$

Therefore, by (A.1), since $u \equiv 0 \in \Psi_2^{-1}(I^r)$,

$$\varphi_2(r) \geq \frac{1}{|r|} \inf_{v\in\Psi_2^{-1}(r)} \Phi(v) \geq \mathcal{K}|r|^{\gamma p/\mathfrak{p}^\star - 1},$$

where $\mathcal{K} = s\mathfrak{K}^{-\gamma p/\mathfrak{p}^\star}/p$. This implies that $\lim\limits_{r\to 0^-} \varphi_2(r) = \infty$, being $\mathfrak{p}^\star > \gamma p$ by $(\mathcal{F})$–(b).

In conclusion, we have proved that

$$\limsup_{r\to 0^-} \varphi_1(r) \leq \varphi_1(0) = \ell^\star < \lim_{r\to 0^-} \varphi_2(r) = \infty. \tag{2.30}$$

This shows that for all integers $k \geq k^\star = 2 + [\ell^\star]$ there exists $r_k < 0$ so close to zero that $\varphi_1(r_k) < \ell^\star + 1/k < k < \varphi_2(r_k)$, that is (A.2) holds. Hence, since $I = \mathbb{R}$, by Theorem Appendix A.1– (ii), Part (a), being $u \equiv 0$ a critical point of J_λ, problem (2.27) admits at least two nontrivial solutions for all

$$\lambda \in \bigcup_{k=k^\star}^{\infty} (\varphi_1(r_k), \varphi_2(r_k)) \supset \bigcup_{k=k^\star}^{\infty} [\ell^\star + 1/k, k] = (\ell^\star, \infty),$$

as claimed. □

We conclude the subsection by noting that $\Phi(u)+\lambda^\star\Psi(u)\geq 0$ for all $u\in\Psi^{-1}(I_0)$ by (2.22). Hence, if $\lambda\leq\lambda^\star$ and $u\in\Psi^{-1}(I_0)$, then

$$J_\lambda(u)=\Phi(u)+\lambda\Psi(u)-\lambda^\star\Psi(u)+\lambda^\star\Psi(u)\geq(\lambda-\lambda^\star)\Psi(u)\geq 0.$$

On the other hand, if $\lambda\geq 0$ and $u\in\Psi^{-1}(I^0)$, then $J_\lambda(u)\geq 0$. Combining both inequalities we get that for all $\lambda\in[0,\lambda^\star]$

$$\inf_{u\in[W_0^{L,p}(\Omega)]^d} J_\lambda(u)=J_\lambda(0)=0.$$

2.5. *The special non–degenerate case when $\gamma=1$*

In this final part of the section we consider the *non–degenerate* problem (1.5). All the assumptions on f, w and M coincide with the hypotheses required for (1.1) in the Introduction and in Section 3.1, with $\gamma=1$. Consequently, *the crucial positive numbers $\lambda_\star$ and $\lambda^\star$ become*

$$\lambda_\star=\frac{s\lambda_1}{1+S_f},\qquad \lambda^\star=\varphi_1(0)\in(\lambda_\star,s\lambda_1),$$

where $\varphi_1(0)$ is given in (2.22), see also (A.1). In this easier setting the proofs of the main Lemmas 2.2–2.5 and of Theorem 2.1 can be reproduced word by word and simplified. In particular, the case $u=0$ in the proof of Lemma 2.2 is redundant, being $M(\tau)\geq s>0$ for all $\tau\in\mathbb{R}_0^+$. Hence, (2.14) immediately gives (2.16) even when $u=0$.

In the recent paper [3] the following quasilinear problem is studied

$$\begin{cases}-\mathrm{div}\mathbf{A}(x,\nabla u)=\lambda\{w(x)|u|^{p-2}u+f(x,u)\} & \text{in } \Omega,\\ \quad u=0 & \text{on } \partial\Omega,\end{cases}\tag{2.31}$$

when $\mathrm{div}\mathbf{A}(x,\nabla u)$ is essentially the p–Laplacian operator. In the special case $\mathrm{div}\mathbf{A}(x,\nabla u)=s\Delta_p u$, problem (2.31) coincides with (1.1), when $d=L=1$ and $M\equiv s$. Theorem 2.1 reduces to Theorem 3.4 of [3].

3. The $p(x)$–polyharmonic Kirchhoff problem

3.1. *Preliminaries*

In this section we extend the results of Section 2.5 to the $p(x)$–polyharmonic Kirchhoff problem (1.6). We begin by recalling some basic results on the variable exponent Lebesgue and Sobolev spaces; see e.g. [19,21]. As before, also here $\Omega\subset\mathbb{R}^n$ *is a bounded domain.* Define for all $h\in C(\overline{\Omega})$

$$h_+=\max_{x\in\overline{\Omega}}h(x)\qquad\text{and}\qquad h_-=\min_{x\in\overline{\Omega}}h(x)$$

and put

$$C_+(\overline{\Omega})=\{h\in C(\overline{\Omega}):h_->1\}.$$

Let h be a fixed function in $C_+(\overline{\Omega})$. The *variable exponent Lebesgue space*

$$L^{h(\cdot)}(\Omega) = \left\{ \psi : \Omega \to \mathbb{R} \text{ measurable} : \int_\Omega |\psi(x)|^{h(x)} dx < \infty \right\}$$

is endowed with the so–called *Luxemburg norm*

$$\|\psi\|_{h(\cdot)} = \inf \left\{ \lambda > 0 : \int_\Omega \left| \frac{\psi(x)}{\lambda} \right|^{h(x)} dx \le 1 \right\}$$

and is a separable, reflexive Banach space; cf. [21, Theorem 2.5 and Corollaries 2.7 and 2.12]. Since here $0 < |\Omega| < \infty$, if $\sigma \in C(\overline{\Omega})$ and $1 \le \sigma \le h$ in Ω, then the embedding $L^{h(\cdot)}(\Omega) \hookrightarrow L^{\sigma(\cdot)}(\Omega)$ is continuous and the norm of the embedding operator does not exceed $|\Omega| + 1$; cf. [21, Theorem 2.8].

Let h' be the function obtained by conjugating the exponent h pointwise, that is $1/h(x) + 1/h'(x) = 1$ for all $x \in \overline{\Omega}$, then h' belongs to $C_+(\overline{\Omega})$ and $L^{h'(\cdot)}(\Omega)$ is the dual space of $L^{h(\cdot)}(\Omega)$, [21, Corollary 2.7]. For any $h_i \in C_+(\overline{\Omega})$, $\psi_i \in L^{h_i(\cdot)}(\Omega)$ for $i = 1, \dots, m$, with $m \ge 1$ and $1 = \sum_{i=1}^m (1/h_i)$, the following *Hölder type inequality* holds

$$\int_\Omega |\psi_1(x) \cdots \psi_m(x)| \, dx \le c_H \|\psi_1\|_{h_1(\cdot)} \cdots \|\psi_m\|_{h_m(\cdot)}, \tag{3.1}$$

where $c_H = 1/h_{1-} + \cdots + 1/h_{m-}$, see [21, Theorem 2.1] for the case $m = 2$.

Let σ be a function in $C(\overline{\Omega})$. An important role in manipulating the generalized Lebesgue–Sobolev spaces is played by the $\sigma(\cdot)$*–modular* of the $L^{\sigma(\cdot)}(\Omega)$ space, which is the convex function $\rho_{\sigma(\cdot)} : L^{\sigma(\cdot)}(\Omega) \to \mathbb{R}$ defined by

$$\rho_{\sigma(\cdot)}(\psi) = \int_\Omega |\psi(x)|^{\sigma(x)} dx.$$

Lemma 3.1 (Theorems 1.3 and 1.4 of [32]). *If ψ, $(\psi_k)_k \subset L^{\sigma(\cdot)}(\Omega)$, with $1 \le \sigma_- \le \sigma_+ < \infty$, then the following relations hold:*

$$\begin{aligned} &\|\psi\|_{\sigma(\cdot)} < 1 \ (= 1; > 1) \Leftrightarrow \rho_{\sigma(\cdot)}(\psi) < 1 \ (= 1; > 1), \\ &\|\psi\|_{\sigma(\cdot)} \ge 1 \quad \Rightarrow \quad \|\psi\|_{\sigma(\cdot)}^{\sigma_-} \le \rho_{\sigma(\cdot)}(\psi) \le \|\psi\|_{\sigma(\cdot)}^{\sigma_+}, \\ &\|\psi\|_{\sigma(\cdot)} \le 1 \quad \Rightarrow \quad \|\psi\|_{\sigma(\cdot)}^{\sigma_+} \le \rho_{\sigma(\cdot)}(\psi) \le \|\psi\|_{\sigma(\cdot)}^{\sigma_-}, \end{aligned} \tag{3.2}$$

and $\|\psi_k - \psi\|_{\sigma(\cdot)} \to 0 \Leftrightarrow \rho_{\sigma(\cdot)}(\psi_k - \psi) \to 0 \Leftrightarrow \psi_k \to \psi$ in measure in Ω and $\rho_{\sigma(\cdot)}(\psi_k) \to \rho_{\sigma(\cdot)}(\psi)$. In particular, $\rho_{\sigma(\cdot)}$ is continuous in $L^{\sigma(\cdot)}(\Omega)$, and if furthermore $\sigma \in C_+(\overline{\Omega})$, then $\rho_{\sigma(\cdot)}$ is weakly lower semi–continuous.

Since we are interested in weighted variable exponent Lebesgue spaces, denoted by ω a generic weight on Ω, we put

$$L^{\sigma(\cdot)}(\Omega, \omega) = \{\psi : \Omega \to \mathbb{R} \,\text{measurable} : \, \omega^{1/\sigma} |\psi| \in L^{\sigma(\cdot)}(\Omega)\},$$

endowed with the norm

$$\|\psi\|_{\sigma(\cdot),\omega} = \|\omega^{1/\sigma} \psi\|_{\sigma(\cdot)}. \tag{3.3}$$

If $p \in C_+(\overline{\Omega})$ and $L = 1, 2, \ldots$, the *variable exponent Sobolev space*

$$W^{L,p(\cdot)}(\Omega) = \Big\{\psi \in L^{p(\cdot)}(\Omega) : D^\alpha\psi \in L^{p(\cdot)}(\Omega) \text{ for all } \alpha \in \mathbb{N}_0^n, \text{ with } |\alpha| \le L\Big\}$$

is endowed with the standard norm

$$\|\psi\|_{W^{L,p(\cdot)}(\Omega)} = \sum_{|\alpha|\le L} \|D^\alpha\psi\|_{p(\cdot)}.$$

From now on we assume that $p \in C_+^{\log}(\overline{\Omega})$, where $C_+^{\log}(\overline{\Omega})$ is the space of all the functions of $C_+(\overline{\Omega})$ which are *logarithmic Hölder continuous*, that is there exists $\mathfrak{K} > 0$ such that

$$|p(x) - p(y)| \le -\frac{\mathfrak{K}}{\log|x-y|} \tag{3.4}$$

for all x, $y \in \Omega$, with $0 < |x - y| \le 1/2$. Indeed, even if the variable exponent Lebesgue and Sobolev spaces have a lot in common with the classical spaces, there are also many fundamental questions left open. For example, it is not known yet, even for "nice" functions p, whether smooth functions are dense in $W^{L,p(\cdot)}(\Omega)$. This is the reason why we assume (3.4).

The space $W_0^{L,p(\cdot)}(\Omega)$ denotes the completion of $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{W^{L,p(\cdot)}(\Omega)}$. As shown in [19, Corollary 11.2.4], the space $W_0^{L,p(\cdot)}(\Omega)$ coincides with the closure in $W^{L,p(\cdot)}(\Omega)$ of the set of all $W^{L,p(\cdot)}(\Omega)$–functions with compact support thanks to (3.4). Moreover $W_0^{L,p(\cdot)}(\Omega)$ is a *separable, uniformly convex, Banach space*, cf. [19, Theorem 8.1.13].

Also in this context it is possible to prove a *Poincaré* type inequality, so that an equivalent norm for the space $W_0^{L,p(\cdot)}(\Omega)$ is given by

$$\|\psi\|_{\mathfrak{D}^{L,p(\cdot)}(\Omega)} = \sum_{|\alpha|=L} \|D^\alpha\psi\|_{p(\cdot)},$$

see [32, Theorem 2.7] and [33, Theorem 4.3].

In what follows, we require that the bounded domain Ω has Lipschitz boundary. Under this assumption, when $L = 2$, as a consequence of the main *Caldéron–Zygmund* result [34, Theorem 6.4], there exists a constant $\kappa_2 = \kappa_2(n, p) > 0$ such that

$$\|\psi\|_{\mathfrak{D}^{2,p(\cdot)}(\Omega)} \le \kappa_2\|\Delta\psi\|_{p(\cdot)} = \kappa_2\|\mathcal{D}_2\psi\|_{p(\cdot)} \quad \text{for all } \psi \in W_0^{2,p(\cdot)}(\Omega), \tag{3.5}$$

where $\mathcal{D}_2$ is defined in (1.3) when $d = 1$, as already noted and used in [1]. For another proof of (3.5) we refer to Theorem 4.4 of [35].

Proposition A.2 of [1] shows that for all $L = 1, 2, \ldots$ there exists a number $\kappa_L = \kappa_L(n, p) > 0$ such that

$$\|\psi\|_{\mathfrak{D}^{L,p(\cdot)}(\Omega)} \le \kappa_L\|\psi\|_{L,p(\cdot)}, \quad \|\psi\|_{L,p(\cdot)} = \begin{cases} \|\mathcal{D}_L\psi\|_{p(\cdot)}, & \text{if } L \text{ is even,} \\ \sum_{i=1}^n \|(\mathcal{D}_L\psi)_i\|_{p(\cdot)}, & \text{if } L \text{ is odd,} \end{cases}$$

for all $\psi \in W_0^{L,p(\cdot)}(\Omega)$, where $\mathcal{D}_L$ is the operator given in (1.3) for $d = 1$. This proves that *the two norms* $\|\cdot\|_{\mathfrak{D}^{L,p(\cdot)}(\Omega)}$ *and* $\|\cdot\|_{L,p(\cdot)}$ *are equivalent.* Hence, also $\left(W_0^{L,p(\cdot)}(\Omega), \|\cdot\|_{L,p(\cdot)}\right)$ is a *reflexive Banach space.* Since we study the variational problem (1.6), when $d = 1$ we actually are interested in the norm

$$\|\psi\| = \begin{cases} \|\mathcal{D}_L\psi\|_{p(\cdot)}, & \text{if } L \text{ is even,} \\ \|\,|\mathcal{D}_L\psi|_n\,\|_{p(\cdot)}, & \text{if } L \text{ is odd.} \end{cases}$$

The two norms $\|\cdot\|_{L,p(\cdot)}$ and $\|\cdot\|$ are exactly the same when L is even. We claim that they are equivalent when L is odd. Let $\psi \in W_0^{L,p(\cdot)}(\Omega)$. First assume that $\|\psi\| > 0$. Then, by definition of the Luxemburg norm, being $|(\mathcal{D}_L\psi)_i| \le |\mathcal{D}_L\psi|_n$ for all $i = 1, \ldots, n$, we have

$$\int_\Omega \left|\frac{(\mathcal{D}_L\psi)_i}{\|\psi\|}\right|^{p(x)} dx \le \int_\Omega \left|\frac{|\mathcal{D}_L\psi|_n}{\|\psi\|}\right|^{p(x)} dx \le 1.$$

Hence $\|(\mathcal{D}_L\psi)_i\|_{p(\cdot)} \le \|\psi\|$. Therefore $\|\psi\|_{L,p(\cdot)} \le n\|\psi\|$. Assume now that $\|\psi\|_{L,p(\cdot)} > 0$. Then $\|(\mathcal{D}_L\psi)_k\|_{p(\cdot)} > 0$ for some $k \in \{1, \ldots, n\}$. Since $1 < p_-$, then $|\mathcal{D}_L\psi|_n^{p(x)} \le n^{p(x)-1}\sum_{i=1}^n |(\mathcal{D}_L\psi)_i|^{p(x)}$. In particular,

$$\begin{aligned}\int_\Omega \left|\frac{|\mathcal{D}_L\psi|_n}{n\|\psi\|_{L,p(\cdot)}}\right|^{p(x)} dx &= \int_\Omega \left|\frac{|\mathcal{D}_L\psi|_n}{n^{1/p'(x)}n^{1/p(x)}\|\psi\|_{L,p(\cdot)}}\right|^{p(x)} dx \le \frac{1}{n}\sum_{i=1}^n \int_\Omega \left|\frac{(\mathcal{D}_L\psi)_i}{\|\psi\|_{L,p(\cdot)}}\right|^{p(x)} dx \\ &\le \frac{1}{n} \sum_{\substack{i=1 \\ \|(\mathcal{D}_L\psi)_i\|_{p(\cdot)}\neq 0}}^n \int_\Omega \left|\frac{(\mathcal{D}_L\psi)_i}{\|(\mathcal{D}_L\psi)_i\|_{p(\cdot)}}\right|^{p(x)} dx \le 1.\end{aligned}$$

Again, by definition of the Luxemburg norm, $\|\psi\| \le n\|\psi\|_{L,p(\cdot)}$. This completes the proof of the claim.

As stated in the Introduction here *either* $n > Lp_+$ *or* $n \le Lp_-$. Hence *the critical variable exponent related to* p is defined for all $x \in \overline{\Omega}$ by the pointwise relation

$$p_L^*(x) = \begin{cases} \dfrac{np(x)}{n - Lp(x)}, & \text{if} \quad n > Lp_+, \\ \infty, & \text{if } 1 \le n \le Lp_-. \end{cases} \tag{3.6}$$

If $n \le Lp_-$, the Sobolev embedding $W_0^{L,p(\cdot)}(\Omega) \hookrightarrow\hookrightarrow L^{h(\cdot)}(\Omega)$ is compact for all $h \in C(\overline{\Omega})$ such that $h \ge 1$ in Ω. Indeed, the embedding $W_0^{L,p(\cdot)}(\Omega) \hookrightarrow W_0^{L,p_-}(\Omega)$ is continuous by Lemma 8.1.8 of [19]. Moreover, since $n \le Lp_-$, the embedding $W_0^{L,p_-}(\Omega) \hookrightarrow\hookrightarrow L^{h_+}(\Omega)$ is compact and in turn also $W_0^{L,p(\cdot)}(\Omega) \hookrightarrow\hookrightarrow L^{h(\cdot)}(\Omega)$ is compact.

If $n > Lp_+$ and $h \in C(\overline{\Omega})$, the embedding $W_0^{L,p(\cdot)}(\Omega) \hookrightarrow L^{h(\cdot)}(\Omega)$ is continuous, whenever $1 \le h(x) \le p_L^*(x)$ for all $x \in \Omega$. A proof of this fact, in the case $L = 1$, is given in [19, Theorem 8.3.1–(a)]. The result then follows by induction on L as in [36, Lemma 5.12]. Moreover, if $1 \le h(x) < p_L^*(x)$ for all $x \in \overline{\Omega}$, then

$W_0^{L,p(\cdot)}(\Omega)$ is compactly embedded into $L^{h(\cdot)}(\Omega)$, as proved in [32, Theorem 2.3], [22, Proposition 3.3] and [18, Theorem 5.7 for $L = 1$].

Since we are interested in the vectorial variational problem (1.6), *from now on we endow the space* $[W_0^{L,p(\cdot)}(\Omega)]^d$ *with the norm*

$$\|u\| = \|\, |\mathcal{D}_L u|_N \,\|_{p(\cdot)},$$

where $|\cdot|_N$ denotes the Euclidean norm in $\mathbb{R}^N$ and $N = d$ when L is even, while $N = dn$ when L is odd. In particular, $\left([W_0^{L,p(\cdot)}(\Omega)]^d, \|\cdot\|\right)$ is a *uniformly convex* Banach space, as proved in Proposition Appendix B.1.

3.2. *The first eigenvalue of* $\boldsymbol{\Delta^L_{p(x)}}$

Taking inspiration from [37], we say that the variable exponent $p \in C_+^{\log}(\overline{\Omega})$ belongs to *the Modular L–Poincaré Inequality Class*, $p \in \mathcal{MP}_L(\Omega)$, if

$$\lambda_1 = \inf_{\substack{u\in[W_0^{L,p(\cdot)}(\Omega)]^d \\ u\neq 0}} \frac{\displaystyle\int_\Omega |\mathcal{D}_L u|_N^{p(x)} dx}{\displaystyle\int_\Omega w(x)|u|_d^{p(x)} dx} > 0, \tag{3.7}$$

where $\mathcal{D}_L$ is given in (1.3).

We point out that there are exponents $p \in C_+^{\log}(\overline{\Omega}) \setminus \mathcal{MP}_L(\Omega)$ even in the case $d = L = 1$ and $w \equiv 1$. For instance, assuming also that there exists an open set $U \subset \Omega$ and a point $x_0 \in U$ such that $p(x_0) < p(x)$ (or $p(x_0) > p(x)$) for all $x \in \partial U$, then by [26, Theorem 3.1] the property (3.7) fails, that is $\lambda_1 = 0$ and p is not of class $\mathcal{MP}_L(\Omega)$.

On the other hand, when $d = L = 1$ and $w \equiv 1$ there are criteria in order that p is of class $\mathcal{MP}_L(\Omega)$, that is p satisfies (3.7). Indeed, by [26, Theorem 3.3] if there exists $l \in \mathbb{R}^n \setminus \{0\}$ such that for all $x \in \Omega$ the function $t \mapsto p(x + tl)$ is monotone for $t \in I_x = \{s \in \mathbb{R} : x + sl \in \Omega\}$, then (3.7) holds. Recently, when $L = d = 1$ and $w \equiv 1$, Theorem 3.3 of [26] has been extended in [28], assuming the existence of a nonnegative function $\xi \in C^1(\overline{\Omega})$, with $|D\xi| > 0$ and $D\xi \cdot Dp \geq 0$ in $\overline{\Omega}$. Another criterium has been proposed in Theorem 1 of [27] again when $L = d = 1$, $w \equiv 1$ and $p \in C^1(\overline{\Omega})$, assuming the existence of $\mathbf{a} : \overline{\Omega} \to \mathbb{R}^n$ such that for all $x \in \overline{\Omega}$

$$\operatorname{div} \mathbf{a}(x) \geq a_0 > 0 \quad \text{and} \quad \mathbf{a}(x) \cdot Dp(x) = 0.$$

The two results of [28] and [27] do not contradict each other but they seem to supplement each other.

Furthermore, [37, Theorem 2.2] says that if (3.7) holds when $L = d = 1$ and $w \equiv 1$, then (3.7) continues to hold for all $w \in L^1_{\text{loc}}(\Omega)$, with $w_- = \operatorname{ess\,inf}_\Omega w(x) > 0$.

Lately, when $d = 1$ and $w \equiv 1$, Theorem 3.1 of [26] has been extended to the $p(x)$–biharmonic operator under Navier boundary conditions. In particular, when p possesses a strict local minimum (or maximum) in Ω and $w_+ = \operatorname{ess\,sup}_\Omega w(x)$ is finite, then

$$\lambda_1 \geq \frac{1}{w_+} \inf_{\psi \in X \setminus \{0\}} \frac{\displaystyle\int_\Omega |\Delta\psi|^{p(x)} dx}{\displaystyle\int_\Omega |\psi|^{p(x)} dx} = 0,$$

where $X = W_0^{1,p(\cdot)}(\Omega) \cap W^{2,p(\cdot)}(\Omega)$. Thus, in principle, (3.7) could fail. As far as we are aware, there are no criteria in order to have $\lambda_1 > 0$, even when $L = 2$, $d = 1$, $w \equiv 1$ and $p(x) \not\equiv p > 1$. In any case we have these useful results, which seem not to be so well known.

Proposition 3.1. *Let $L \in \{1, 2\}$. If $\lambda_1 > 0$ when $d = 1$, then $\lambda_1 > 0$ for all $d \in \mathbb{N}$.*

Proof. Consider first the case $L = 1$. Clearly, by density, for $d = 1$,

$$\lambda_1 = \inf_{\substack{\psi \in C_0^\infty(\Omega) \\ \psi \neq 0}} \frac{\displaystyle\int_\Omega |D\psi|_n^{p(x)} dx}{\displaystyle\int_\Omega w(x)|\psi|^{p(x)} dx} > 0.$$

Then for all $\varphi = (\varphi_1, \ldots, \varphi_d) \in [C_0^\infty(\Omega)]^d \setminus \{0\}$

$$\begin{aligned}\frac{\displaystyle\int_\Omega |D\varphi|_{dn}^{p(x)} dx}{\displaystyle\int_\Omega w(x)|\varphi|_d^{p(x)} dx} &\geq \frac{\displaystyle\int_\Omega d^{-p(x)/2} \left(\sum_{i=1}^d |D\varphi_i|_n\right)^{p(x)} dx}{\displaystyle d^{p_+-1} \sum_{i=1}^d \int_\Omega w(x)|\varphi_i|^{p(x)} dx} \\ &\geq d^{1-3p_+/2} \cdot \frac{\displaystyle\sum_{i=1}^d \int_\Omega |D\varphi_i|_n^{p(x)} dx}{\displaystyle\sum_{i=1}^d \int_\Omega w(x)|\varphi_i|^{p(x)} dx} \geq d^{1-3p_+/2} \lambda_1,\end{aligned}$$

that is (3.7) holds for all $d \geq 2$ when $L = 1$.

Similarly, when $L = 2$, by assumption

$$\lambda_1 = \inf_{\substack{\psi \in C_0^\infty(\Omega) \\ \psi \neq 0}} \frac{\displaystyle\int_\Omega |\Delta\psi|^{p(x)} dx}{\displaystyle\int_\Omega w(x)|\psi|^{p(x)} dx} > 0.$$

Hence, for all $\varphi = (\varphi_1, \dots, \varphi_d) \in [C_0^\infty(\Omega)]^d \setminus \{0\}$

$$\frac{\displaystyle\int_\Omega |\Delta\varphi|_d^{p(x)} dx}{\displaystyle\int_\Omega w(x)|\varphi|_d^{p(x)} dx} \geq \frac{\displaystyle\int_\Omega d^{-p(x)/2}\left(\sum_{i=1}^d |\Delta\varphi_i|\right)^{p(x)} dx}{\displaystyle d^{p_+-1}\sum_{i=1}^d \int_\Omega w(x)|\varphi_i|^{p(x)} dx}$$

$$\geq d^{1-3p_+/2} \cdot \frac{\displaystyle\sum_{i=1}^d \int_\Omega |\Delta\varphi_i|^{p(x)} dx}{\displaystyle\sum_{i=1}^d \int_\Omega w(x)|\varphi_i|^{p(x)} dx} \geq d^{1-3p_+/2}\lambda_1,$$

so that (3.7) holds for all $d \geq 2$ when $L = 2$. □

Proposition 3.2. *Assume that $w_- > 0$. If (3.7) holds for $L \in \{1, 2\}$, then it holds for all $L \in \mathbb{N}$.*

Proof. Let us denote for simplicity the number in (3.7) by $\lambda_{1,1}$ if $L = 1$ and $\lambda_{1,2}$ if $L = 2$. By density,

$$\lambda_{1,1} = \inf_{\substack{\varphi\in[C_0^\infty(\Omega)]^d \\ \varphi\neq 0}} \frac{\displaystyle\int_\Omega |D\varphi|_{dn}^{p(x)} dx}{\displaystyle\int_\Omega w(x)|\varphi|^{p(x)} dx} > 0, \quad \lambda_{1,2} = \inf_{\substack{\varphi\in[C_0^\infty(\Omega)]^d \\ \varphi\neq 0}} \frac{\displaystyle\int_\Omega |\Delta\varphi|_d^{p(x)} dx}{\displaystyle\int_\Omega w(x)|\varphi|^{p(x)} dx} > 0.$$

Now, $\mathcal{D}_3\varphi = D\Delta\varphi$ by (1.3). Hence, for any $\varphi \in [C_0^\infty(\Omega)]^d \setminus \{0\}$

$$\frac{\displaystyle\int_\Omega |\mathcal{D}_3\varphi|_{dn}^{p(x)} dx}{\displaystyle\int_\Omega w(x)|\varphi|_d^{p(x)} dx} = \frac{\displaystyle\int_\Omega |D\Delta\varphi|_{dn}^{p(x)} dx}{\displaystyle\int_\Omega w(x)|\Delta\varphi|_d^{p(x)} dx} \cdot \frac{\displaystyle\int_\Omega w(x)|\Delta\varphi|_d^{p(x)} dx}{\displaystyle\int_\Omega w(x)|\varphi|_d^{p(x)} dx} \geq \lambda_{1,1}w_-\lambda_{1,2}.$$

In other words, $\lambda_{1,3} \geq \lambda_{1,1}w_-\lambda_{1,2}$.

Similarly, $\mathcal{D}_4\varphi = \Delta^2\varphi$ by (1.3) and for any $\varphi \in [C_0^\infty(\Omega)]^d \setminus \{0\}$

$$\frac{\displaystyle\int_\Omega |\mathcal{D}_4\varphi|_d^{p(x)} dx}{\displaystyle\int_\Omega w(x)|\varphi|_d^{p(x)} dx} = \frac{\displaystyle\int_\Omega |\Delta(\Delta\varphi)|_d^{p(x)} dx}{\displaystyle\int_\Omega w(x)|\Delta\varphi|_d^{p(x)} dx} \cdot \frac{\displaystyle\int_\Omega w(x)|\Delta\varphi|_d^{p(x)} dx}{\displaystyle\int_\Omega w(x)|\varphi|_d^{p(x)} dx} \geq w_-\lambda_{1,2}^2,$$

that is $\lambda_{1,4} \geq w_-\lambda_{1,2}^2$. In conclusion, by induction,

$$\lambda_1 = \lambda_{1,L} \geq \begin{cases} w_-^{j-1}\lambda_{1,2}^j, & L = 2j, \\ \lambda_{1,1}(w_-\lambda_{1,2})^{j-1}, & L = 2j-1, \end{cases} \quad \text{for } j = 2, 3 \dots.$$

In particular, $\lambda_1 > 0$ for all $L \in \mathbb{N}$, as required. □

3.3. *The main existence result for* (1.6)

We finally turn to problem (1.6), recalling the assumptions required. As in (1.4), the weight w is supposed to be *positive a.e. in* Ω *and of class* $L^{\varpi}(\Omega)$, *with*

$$\varpi > \frac{n}{n - [n - Lp_-]^+} \tag{3.8}$$

replacing (1.4), and the variable exponent p is assumed also of class $\mathcal{MP}_L(\Omega)$.

The Kirchhoff function M verifies the assumption $(\mathcal{M})$, with $\gamma = 1$, that is (1.6) is non–degenerate. The Dirichlet functional $\mathscr{I}_L$ is defined in (1.7). The nonlinearity f verifies the foreword in $(\mathcal{F})$, condition $(\mathcal{F})$–$(c)'$ of Section 2.4, while (a) and (b) are replaced by

$(a)'$ *There exist* $q \in C_+(\overline{\Omega})$, *with* $1 < q_+ < p_-$, *and* $C_f > 0$ *such that*

$$|f(x, v)| \le C_f w(x)\big(1 + |v|^{q(x)-1}\big) \text{ for a.a. } x \in \Omega \text{ and all } v \in \mathbb{R}^d.$$

$(b)'$ *There exists* $\mathfrak{p}^\star \in C_+(\overline{\Omega})$, *with* $p_+ < \mathfrak{p}^\star_- \le \mathfrak{p}^\star_+ < (p^*_L)_-/\varpi'$, *such that*

$$\limsup_{|v|\to 0} \frac{|f(x, v) \cdot v|}{w(x)|v|^{\mathfrak{p}^\star(x)}} < \infty, \quad \text{uniformly a.e. in } \Omega.$$

Also in this setting we have the analogue of Proposition 2.2, that is

Proposition 3.3. *Assume that* $(\mathcal{F})$–$(a)'$ *and* $(b)'$ *hold. Then* $f(x, 0) = 0$ *for a.a.* $x \in \Omega$,

$$0 < S_f = \operatorname*{ess\,sup}_{v\neq 0,\, x\in\Omega} \frac{|f(x, v) \cdot v|}{w(x)|v|^{p(x)}} < \infty, \quad 0 < \operatorname*{ess\,sup}_{v\neq 0,\, x\in\Omega} \frac{|F(x, v)|}{w(x)|v|^{p(x)}} \le \frac{S_f}{p_-}. \tag{3.9}$$

Moreover, there exists $K > 0$ *such that*

$$|F(x, v)| \le K \frac{w(x)}{\mathfrak{p}^\star(x)} |v|^{\mathfrak{p}^\star(x)} \tag{3.10}$$

for a.a. $x \in \Omega$ *and all* $v \in \mathbb{R}^d$.

The energy functional $J_\lambda : [W_0^{L,p(\cdot)}(\Omega)]^d \to \mathbb{R}$ associated to (1.6) is given by $J_\lambda(u) = \Phi(u) + \lambda\Psi(u)$, where now $\Phi(u) = \mathscr{M}(\mathscr{I}_L(u))$, with $\mathscr{I}_L(u)$ defined in (1.7) and $\Psi = \Psi_2$, where as usual

$$\Psi_2(u) = -\int_\Omega F(x, u(x))dx.$$

The dual space of $[W_0^{L,p(\cdot)}(\Omega)]^d$ is denoted by $\Big([W_0^{L,p(\cdot)}(\Omega)]^d\Big)^\star$. We are now proving a result similar to Lemma 2.2.

Lemma 3.2. *The functional* Φ *is convex, weakly lower semi–continuous in* $[W_0^{L,p(\cdot)}(\Omega)]^d$ *and of class* $C^1([W_0^{L,p(\cdot)}(\Omega)]^d)$.

Moreover, $\Phi' : [W_0^{L,p(\cdot)}(\Omega)]^d \to \left([W_0^{L,p(\cdot)}(\Omega)]^d\right)^\star$ *verifies the* $(\mathscr{S}_+)$ *condition, i.e. if* $u_k \rightharpoonup u$ *in* $[W_0^{L,p(\cdot)}(\Omega)]^d$ *and*

$$\limsup_{k\to\infty} M(\mathscr{I}_L(u_k)) \int_\Omega |\mathcal{D}_L u_k|^{p(x)-2}\mathcal{D}_L u_k \cdot (\mathcal{D}_L u_k - \mathcal{D}_L u)dx \le 0, \tag{3.11}$$

then $u_k \to u$ *in* $[W_0^{L,p(\cdot)}(\Omega)]^d$.

Proof. A simple calculation shows that Φ is convex in $[W_0^{L,p(\cdot)}(\Omega)]^d$, being $\mathscr{I}_L$ convex and M non–negative and non–decreasing by $(\mathcal{M})$. Moreover, Φ is Gâteaux differentiable in $[W_0^{L,p(\cdot)}(\Omega)]^d$ and for all $u, v \in [W_0^{L,p(\cdot)}(\Omega)]^d$ it results

$$\langle \Phi'(u), v\rangle = M(\mathscr{I}_L(u)) \int_\Omega |\mathcal{D}_L u|^{p(x)-2}\mathcal{D}_L u \cdot \mathcal{D}_L v\, dx.$$

Now, let u, $(u_k) \subset [W_0^{L,p(\cdot)}(\Omega)]^d$ be such that $u_k \to u$ as $k \to \infty$. We claim that

$$\|\Phi'(u_k) - \Phi'(u)\|_\star = \sup_{\substack{v\in[W_0^{L,p(\cdot)}(\Omega)]^d\\ \|v\|=1}} |\langle \Phi'(u_k) - \Phi'(u), v\rangle| = o(1) \quad \text{as } k \to \infty.$$

Put $\mathscr{R}(u_k, u) = \big\| M(\mathscr{I}_L(u_k))|\mathcal{D}_L u_k|^{p(x)-2}\mathcal{D}_L u_k - M(\mathscr{I}_L(u))|\mathcal{D}_L u|^{p(x)-2}\mathcal{D}_L u\big\|_{N,p'(\cdot)}$. By the Hölder inequality (3.1)

$$\big|\langle \Phi'(u_k) - \Phi'(u), v\rangle\big| \le c_H \mathscr{R}(u_k, u)\|\mathcal{D}_L v\|_{N,p(\cdot)}.$$

Hence

$$\|\Phi'(u_k) - \Phi'(u)\|_\star \le c_H \mathscr{R}(u_k, u). \tag{3.12}$$

Let $(u_{k_j})_j$ be a subsequence of $(u_k)_k$. Clearly, $u_{k_j} \to u$ in $[W_0^{L,p(\cdot)}(\Omega)]^d$ and so $\mathcal{D}_L u_{k_j} \to \mathcal{D}_L u$ in $[L^{p(\cdot)}(\Omega)]^N$ as $j \to \infty$, where as usual $N = d$ if L is even and $N = dn$ if L is odd. By Lemma Appendix B.1, with $m = N$, $\sigma = p$ and $\omega \equiv 1$, there exist a subsequence of $(u_{k_j})_j$, still denoted by $(u_{k_j})_j$, and $h \in L^{p(\cdot)}(\Omega)$ such that for a.a. $x \in \Omega$

$$\mathcal{D}_L u_{k_j}(x) \to \mathcal{D}_L u(x) \quad \text{as } j \to \infty \quad \text{and} \quad |\mathcal{D}_L u_{k_j}(x)| \le h(x) \quad \text{for all } j \in \mathbb{N}.$$

Hence,

$$\begin{aligned}
&\left| M(\mathscr{I}_L(u_{k_j}))|\mathcal{D}_L u_{k_j}|^{p(x)-2}\mathcal{D}_L u_{k_j} - M(\mathscr{I}_L(u))|\mathcal{D}_L u|^{p(x)-2}\mathcal{D}_L u\right|^{p'(x)}\\
&\quad \le 2^{p'(x)-1}\left\{\left[M(\mathscr{I}_L(u_{k_j}))|\mathcal{D}_L u_{k_j}|^{p(x)-1}\right]^{p'(x)} + \left[M(\mathscr{I}_L(u))|\mathcal{D}_L u|^{p(x)-1}\right]^{p'(x)}\right\}\\
&\quad \le (2K)^{p'(x)} h^{p(x)} \in L^1(\Omega),
\end{aligned}$$

where $K = \sup_j M(\mathscr{I}_L(u_{k_j})) < \infty$, being $(u_{k_j})_j$ convergent and so bounded in $[W_0^{L,p(\cdot)}(\Omega)]^d$. In particular, $M(\mathscr{I}_L(u_{k_j})) \to M(\mathscr{I}_L(u))$ by $(\mathcal{M})$. Furthermore,

$|\mathcal{D}_L u_{k_j}|^{p(x)-2}\mathcal{D}_L u_{k_j} \to |\mathcal{D}_L u|^{p(x)-2}\mathcal{D}_L u$ a.e. in Ω. Hence

$$\begin{aligned}\Big|M(\mathscr{I}_L(u_{k_j}))|\mathcal{D}_L u_{k_j}|^{p(x)-2}\mathcal{D}_L u_{k_j} - M(\mathscr{I}_L(u))|\mathcal{D}_L u|^{p(x)-2}\mathcal{D}_L u\Big| \\ \le K\Big||\mathcal{D}_L u_{k_j}|^{p(x)-2}\mathcal{D}_L u_{k_j} - |\mathcal{D}_L u|^{p(x)-2}\mathcal{D}_L u\Big| \\ + \big|M(\mathscr{I}_L(u_{k_j})) - M(\mathscr{I}_L(u))\big| \cdot |\mathcal{D}_L u|^{p(x)-1} \to 0\end{aligned}$$

a.e. in Ω as $j \to \infty$. Thus, applying the Lebesgue dominated convergence theorem, we obtain that the entire sequence $(u_k)_k$ is such that $\mathscr{R}(u_k, u) \to 0$ as $k \to \infty$, which implies the claim by (3.12). In conclusion, Φ is of class $C^1([W_0^{L,p(\cdot)}(\Omega)]^d)$, as claimed. In particular, Φ is weakly lower semi–continuous in $[W_0^{L,p(\cdot)}(\Omega)]^d$ by Corollary 3.9 of [30].

Let $u, (u_k)_k \subset [W_0^{L,p(\cdot)}(\Omega)]^d$ be such that $u_k \rightharpoonup u$ in $[W_0^{L,p(\cdot)}(\Omega)]^d$ and (3.11) hold. Then

$$\lim_{k\to\infty} M(\mathscr{I}_L(u)) \int_\Omega |\mathcal{D}_L u|^{p(x)-2}\mathcal{D}_L u \cdot \mathcal{D}_L(u_k - u)dx = 0, \tag{3.13}$$

being $|\mathcal{D}_L u|^{p(x)-2}\mathcal{D}_L u \in [L^{p'(\cdot)}(\Omega)]^N$. Hence (3.11) is equivalent to

$$\limsup_{k\to\infty} \int_\Omega \mathscr{L}(u_k, u) \cdot \mathcal{D}_L(u_k - u)dx \le 0,$$

where $\mathscr{L}(u_k, u) = M(\mathscr{I}_L(u_k))|\mathcal{D}_L u_k|^{p(x)-2}\mathcal{D}_L u_k - M(\mathscr{I}_L(u))|\mathcal{D}_L u|^{p(x)-2}\mathcal{D}_L u$. By convexity

$$\int_\Omega \mathscr{L}(u_k, u) \cdot \mathcal{D}_L(u_k - u)dx \ge 0,$$

therefore

$$\lim_{k\to\infty} \int_\Omega \mathscr{L}(u_k, u) \cdot \mathcal{D}_L(u_k - u)dx = 0.$$

This implies $\lim_{k\to\infty} M(\mathscr{I}_L(u_k)) \int_\Omega |\mathcal{D}_L u_k|^{p(x)-2}\mathcal{D}_L u_k \cdot \mathcal{D}_L(u_k - u)dx = 0$ by (3.13), and in turn

$$\lim_{k\to\infty} \int_\Omega |\mathcal{D}_L u_k|^{p(x)-2}\mathcal{D}_L u_k \cdot \mathcal{D}_L(u_k - u)dx = 0, \tag{3.14}$$

being $M(\mathscr{I}_L(u_k)) \ge s > 0$ for all $k \in \mathbb{N}$, since here $\gamma = 1$ in $(\mathcal{M})$.

Clearly $\mathscr{I}_L$ is of class C^1 and convex in the Banach space $[W_0^{L,p(\cdot)}(\Omega)]^d$, so that $\mathscr{I}_L(u) \le \liminf_{k\to\infty} \mathscr{I}_L(u_k)$ by the weak lower semi–continuity of $\mathscr{I}_L$ in $[W_0^{L,p(\cdot)}(\Omega)]^d$. By the convexity of $\mathscr{I}_L$ for all k

$$\mathscr{I}_L(u) + \int_\Omega |\mathcal{D}_L u_k|^{p(x)-2}\mathcal{D}_L u_k \cdot \mathcal{D}_L(u_k - u)dx \ge \mathscr{I}_L(u_k),$$

so that $\mathscr{I}_L(u) \ge \limsup_{k\to\infty} \mathscr{I}_L(u_k)$ by (3.14). In conclusion,

$$\lim_{k\to\infty} \mathscr{I}_L(u_k) = \mathscr{I}_L(u). \tag{3.15}$$

Furthermore, by (3.14)

$$\int_{\Omega} (|\mathcal{D}_L u_k|^{p(x)-2} \mathcal{D}_L u_k - |\mathcal{D}_L u|^{p(x)-2} \mathcal{D}_L u) \cdot \mathcal{D}_L(u_k - u) dx \to 0 \quad \text{as } k \to \infty,$$

since $u_k \rightharpoonup u$ in $[W_0^{L,p(\cdot)}(\Omega)]^d$. Hence $(|\mathcal{D}_L u_k|^{p(x)-2} \mathcal{D}_L u_k - |\mathcal{D}_L u|^{p(x)-2} \mathcal{D}_L u) \cdot \mathcal{D}_L(u_k - u) \geq 0$ converges to 0 in $L^1(\Omega)$, and so, up to a subsequence,

$$(|\mathcal{D}_L u_{k_j}|^{p(x)-2} \mathcal{D}_L u_{k_j} - |\mathcal{D}_L u|^{p(x)-2} \mathcal{D}_L u) \cdot \mathcal{D}_L(u_{k_j} - u) \to 0$$

a.e. in Ω. Lemma 3 of [38] implies that $\mathcal{D}_L u_{k_j}$ converges to $\mathcal{D}_L u$ a.e. in Ω, and in turn $|\mathcal{D}_L u_{k_j}|^{p(x)}$ converges to $|\mathcal{D}_L u|^{p(x)}$ a.e. in Ω.

Consider the sequence $(g_j)_j$ in $L^1(\Omega)$ defined pointwise by

$$g_j(x) = \frac{1}{p(x)} \left\{ \frac{|\mathcal{D}_L u_{k_j}|^{p(x)} + |\mathcal{D}_L u|^{p(x)}}{2} - \left| \frac{\mathcal{D}_L u_{k_j} - \mathcal{D}_L u}{2} \right|^{p(x)} \right\}.$$

By convexity $g_j \geq 0$ and $g_j(x) \to |\mathcal{D}_L u(x)|^{p(x)}/p(x)$ for a.a. $x \in \Omega$. Therefore, by the Fatou lemma and (3.15) we have

$$\begin{aligned} \mathscr{I}_L(u) &\leq \liminf_{j\to\infty} \int_{\Omega} g_j dx = \mathscr{I}_L(u) - \limsup_{j\to\infty} \int_{\Omega} \frac{1}{p(x)} \left| \frac{\mathcal{D}_L u_{k_j} - \mathcal{D}_L u}{2} \right|^{p(x)} dx \\ &\leq \mathscr{I}_L(u) - \frac{1}{p_+ 2^{p_+}} \limsup_{j\to\infty} \rho_{p(\cdot)}(\mathcal{D}_L u_{k_j} - \mathcal{D}_L u). \end{aligned}$$

Hence, $\limsup_{j\to\infty} \rho_{p(\cdot)}(\mathcal{D}_L u_{k_j} - \mathcal{D}_L u) = 0$, that is $\lim_{j\to\infty} \|u_{k_j} - u\| = 0$ by Lemma 3.1. In conclusion, the entire sequence $u_k \to u$, since $u_k \rightharpoonup u$ as $k \to \infty$ in $[W_0^{L,p(\cdot)}(\Omega)]^d$. This completes the proof. $\square$

As in Section 2.1, if the embedding operator $i : [W_0^{L,p(\cdot)}(\Omega)]^d \to [L^{h(\cdot)}(\Omega,\omega)]^d$ is continuous, where $h \in C(\overline{\Omega})$, $h \geq 1$, and ω is a weight, we denote by $\mathcal{S}_{d,h(\cdot),\omega} > 0$ the best constant such that $\|u\|_{h(\cdot),\omega} \leq \mathcal{S}_{d,h(\cdot),\omega}\|u\|$ for all $u \in [W_0^{L,p(\cdot)}(\Omega)]^d$. Again $\mathcal{S}_{d,h(\cdot),\omega}$ is the operator norm of i. If $d = 1$ and $\omega \equiv 1$, we briefly write $\mathcal{S}_{h(\cdot)}$. Furthermore, if $p_L^* \equiv \infty$ the symbol p_L^*/ϖ' is ∞.

Lemma 3.3. *Let $\sigma \in C(\overline{\Omega})$, with $\sigma_- \geq 1$. If $\sigma(x) \leq p(x)$ for all $x \in \Omega$, then the embedding $[W_0^{L,p(\cdot)}(\Omega)]^d \hookrightarrow\hookrightarrow [L^{\sigma(\cdot)}(\Omega,w)]^d$ is compact.*

Proof. The embedding $W_0^{L,p(\cdot)}(\Omega) \hookrightarrow\hookrightarrow L^{\varpi'\sigma}(\Omega)$ is compact, being $\varpi'\sigma(x) < p_L^*(x)$ for all $x \in \overline{\Omega}$ by (3.6) and (3.8). Furthermore, $L^{\varpi'\sigma(\cdot)}(\Omega)$ is continuously embedded in $L^{\sigma(\cdot)}(\Omega,w)$ by the classical Hölder inequality and (3.8). Hence, as in the proof of Lemma 2.1–(i), also $[W_0^{L,p}(\Omega)]^d \hookrightarrow\hookrightarrow [L^{\sigma(\cdot)}(\Omega,w)]^d$. $\square$

Lemma 3.4. *Let $(\mathcal{F})$–$(a)'$ hold. Then $\Psi_2' : [W_0^{L,p(\cdot)}(\Omega)]^d \to \left([W_0^{L,p(\cdot)}(\Omega)]^d\right)^\star$ is a compact operator and Ψ_2 is sequentially weakly continuous in $[W_0^{L,p(\cdot)}(\Omega)]^d$.*

Proof. Of course, $\langle \Psi_2'(u), v\rangle = -\int_\Omega f(x,u)\cdot v\,dx$ for all $u, v \in [W_0^{L,p}(\Omega)]^d$. Since Ψ_2' is continuous and $[W_0^{L,p(\cdot)}(\Omega)]^d$ is reflexive, it is enough to show that if $(u_k)_k$, u are in $[W_0^{L,p(\cdot)}(\Omega)]^d$ and $u_k \rightharpoonup u$ in $[W_0^{L,p(\cdot)}(\Omega)]^d$, then $\|\Psi_2'(u_k) - \Psi_2'(u)\|_\star \to 0$ as $k \to \infty$. To this aim, fix $(u_k)_k \subset [W_0^{L,p(\cdot)}(\Omega)]^d$, with $u_k \rightharpoonup u$ in $[W_0^{L,p(\cdot)}(\Omega)]^d$.

First, $u_k \to u$ in $[L^{q(\cdot)}(\Omega, w)]^d$ by Lemma 3.3–(i). Thus, $\mathcal{N}_f(u_k) \to \mathcal{N}_f(u)$ as $k \to \infty$ in $[L^{q'(\cdot)}(\Omega, w^{1/(1-q)})]^d$ by Lemma Appendix B.2. Finally, by Hölder's inequality for all $v \in [W_0^{L,p(\cdot)}(\Omega)]^d$, with $\|v\| = 1$, we have

$$\begin{aligned}
|\langle \Psi_2'(u_k) - \Psi_2'(u), v\rangle| &\le \int_\Omega w(x)^{-1/q(x)} |\mathcal{N}_f(u_k) - \mathcal{N}_f(u)| w^{1/q(x)} |v| dx \\
&\le c_H \| w^{-1/q} |\mathcal{N}_f(u_k) - \mathcal{N}_f(u)|_d \|_{q'(\cdot)} \| w^{1/q} |v|_d \|_{q(\cdot)} \\
&= c_H \|\mathcal{N}_f(u_k) - \mathcal{N}_f(u)\|_{d,q'(\cdot),w^{1/(1-q)}} \|v\|_{d,q(\cdot),w} \\
&\le c_H \mathcal{S}_{d,q(\cdot),w} \|\mathcal{N}_f(u_k) - \mathcal{N}_f(u)\|_{d,q'(\cdot),w^{1/(1-q)}}.
\end{aligned}$$

Thus, $\|\Psi_2'(u_k) - \Psi_2'(u)\|_\star \to 0$ as $k \to \infty$, that is Ψ_2' is compact.

Now, since Ψ_2' is compact, then Ψ_2 is sequentially weakly continuous by Corollary 41.9 of [31], being $[W_0^{L,p(\cdot)}(\Omega)]^d$ reflexive. □

Theorem 3.1. *Let* $(\mathcal{F})$*–*$(a)'$*,* $(b)'$ *hold and let* $\ell_\star = \dfrac{p_- s\lambda_1}{p_+ S_f}$.

(i) *If* $\lambda \in [0, \ell_\star)$*, then* (1.6) *has only the trivial solution.*
(ii) *If also* $(\mathcal{F})$*–*$(c)'$ *holds, then there exists* $\ell^\star \ge \ell_\star$ *such that* (1.6) *admits at least two nontrivial solutions for all* $\lambda \in (\ell^\star, \infty)$*.*

Proof. The part (i) of the statement is proved by using a similar argument produced for the proof of Theorem 2.1–(i), namely if u is a weak solution of (1.6) we have

$$\begin{aligned}
sp_-\lambda_1 \mathscr{I}_L(u) &\le \lambda_1 M(\mathscr{I}_L(u)) \int_\Omega |\mathcal{D}_L u|^{p(x)} dx \le \lambda_1 \lambda \int_\Omega |f(x,u)\cdot u|\,dx \\
&\le \lambda_1 \lambda S_f \int_\Omega w(x)|u|^{p(x)} dx \le p_+ \lambda S_f \mathscr{I}_L(u),
\end{aligned}$$

by (3.7) and (3.9). Thus, if $u \not\equiv 0$, then necessarily $\lambda \ge \ell_\star$.

In order to prove (ii), we first show that J_λ is coercive for every $\lambda \in \mathbb{R}$. Indeed, as shown in the proof of Lemma 2.5, for all $u \in [W_0^{L,p(\cdot)}(\Omega)]^d$, with $\|u\| \ge 1$,

$$\begin{aligned}
J_\lambda(u) &\ge s\mathscr{I}_L(u) - |\lambda| \int_\Omega |F(x,u)| dx \\
&\ge \frac{s}{p_+} \int_\Omega |\mathcal{D}_L u|^{p(x)} dx - |\lambda| \int_\Omega \int_0^1 C_f w(x) \left(1 + |tu|^{q(x)-1}\right) |u| dt\, dx \qquad (3.16) \\
&\ge \frac{s}{p_+} \|u\|^{p_-} - |\lambda| C_f \int_\Omega w(x)|u| dx - \frac{|\lambda| C_f}{q_-} \int_\Omega w(x)|u|^{q(x)} dx.
\end{aligned}$$

Now, by Hölder's inequality, Lemma 3.1 and Lemma 3.3

$$\begin{aligned}\int_\Omega w(x)|u|^{q(x)}dx &\le \left(\int_\Omega w(x)^\varpi dx\right)^{1/\varpi}\left(\int_\Omega |u|^{\varpi' q(x)}dx\right)^{1/\varpi'}\\ &\le \|w\|_\varpi \max\left\{\|u\|^{q_-}_{d,\varpi' q(\cdot)}, \|u\|^{q_+}_{d,\varpi' q(\cdot)}\right\}\\ &\le \|w\|_\varpi \max\left\{\mathcal{S}^{q_-}_{d,\varpi' q(\cdot)}\|u\|^{q_-}, \mathcal{S}^{q_+}_{d,\varpi' q(\cdot)}\|u\|^{q_+}\right\}.\end{aligned}$$

Hence, for all $u \in [W_0^{L,p(\cdot)}(\Omega)]^d$, with $\|u\| \ge 1$, by (3.16)

$$J_\lambda(u) \ge \frac{s}{p_+}\|u\|^{p_-} - |\lambda| C_f \mathcal{S}_{d,1,w}\|u\| - |\lambda| C \|u\|^{q_+},$$

where $C = C_f \|w\|_\varpi \max\left\{\mathcal{S}^{q_-}_{d,\varpi' q(\cdot)}, \mathcal{S}^{q_+}_{d,\varpi' q(\cdot)}\right\}/q_-$. Since $1 < q_+ < p_-$, this shows the claim by $(\mathcal{F})$–$(a)'$. Thus, here $I = \mathbb{R}$.

The function $u_0 \in [C_0^\infty(\Omega)]^d$ constructed by using $(\mathcal{F})$–$(c)'$ in the proof of Theorem 2.2 is also in $[W_0^{L,p(\cdot)}(\Omega)]^d$ and such that $\Psi_2(u_0) < 0$. Hence, also in this setting the crucial number

$$\ell^\star = \varphi_1(0) = \inf_{u \in \Psi_2^{-1}(I_0)} -\frac{\Phi(u)}{\Psi_2(u)}, \quad I_0 = (-\infty, 0), \tag{3.17}$$

is well defined, so that again (2.29) continues to hold. Furthermore, by (3.7) and (3.9)

$$\frac{\Phi(u)}{|\Psi_2(u)|} \ge \frac{s\displaystyle\int_\Omega |\mathcal{D}_L u|^{p(x)}dx}{p_+\displaystyle\int_\Omega |F(x,u)|dx} \ge \frac{p_- s\displaystyle\int_\Omega |\mathcal{D}_L u|^{p(x)}dx}{p_+ S_f\displaystyle\int_\Omega w(x)|u|^{p(x)}dx} \ge \frac{p_- s\lambda_1}{p_+ S_f} = \ell_\star$$

for all $u \in \Psi_2^{-1}(I_0)$. Thus, $\ell^\star \ge \ell_\star > 0$ by (3.17).

By (3.10), Hölder's inequality (3.1), Lemma 2.1 of [20] and the continuity of the embedding $[W_0^{L,p(\cdot)}(\Omega)]^d \hookrightarrow [L^{p_L^*(\cdot)}(\Omega)]^d$, we have for every $u \in [W_0^{L,p(\cdot)}(\Omega)]^d$

$$\begin{aligned}|\Psi_2(u)| &\le \int_\Omega |F(x,u)|\,dx \le \frac{K}{\mathfrak{p}^\star_-}\int_\Omega w(x)|u|^{\mathfrak{p}^\star(x)}dx \le c\,\||u|^{\mathfrak{p}^\star(x)}\|_{p_L^*(\cdot)/\mathfrak{p}^\star(\cdot)}\\ &\le c\max\left\{\|u\|^{\mathfrak{p}^\star_+}_{d,p_L^*(\cdot)}, \|u\|^{\mathfrak{p}^\star_-}_{d,p_L^*(\cdot)}\right\} \le \mathfrak{K}\max\{\|u\|^{\mathfrak{p}^\star_+}, \|u\|^{\mathfrak{p}^\star_-}\},\end{aligned} \tag{3.18}$$

where $c = c_H K \|1\|_{\wp(\cdot)}\|w\|_\varpi/\mathfrak{p}^\star_-$, $\mathfrak{K} = c\max\left\{\mathcal{S}^{\mathfrak{p}^\star_+}_{d,p_L^*(\cdot)}, \mathcal{S}^{\mathfrak{p}^\star_-}_{d,p_L^*(\cdot)}\right\}$, and

$$1 < \wp(x) = \begin{cases}\dfrac{\varpi' p_L^*(x)}{p_L^*(x) - \mathfrak{p}^\star(x)\varpi'}, & \text{if } n > Lp_+,\\ \varpi', & \text{if } n \le Lp_-.\end{cases}$$

Taken $r<0$ and $v \in \Psi_2^{-1}(r)$, we obtain by (3.2), (3.18), and $(\mathcal{M})$, with $\gamma = 1$,

$$
\begin{aligned}
|r| = |\Psi_2(v)| &\le \mathfrak{K} \max\left\{ \left(p_+\!\int_\Omega \frac{|\mathcal{D}_L u|^{p(x)}}{p(x)}dx\right)^{\mathfrak{p}_+^\star/p_-}, \left(p_+\!\int_\Omega \frac{|\mathcal{D}_L u|^{p(x)}}{p(x)}dx\right)^{\mathfrak{p}_-^\star/p_+} \right\} \\
&\le (p_+)^{\mathfrak{p}_+^\star/p_-} \mathfrak{K} \max\left\{ \left(\frac{\mathscr{M}(\mathscr{I}_L(u))}{s}\right)^{\mathfrak{p}_+^\star/p_-}, \left(\frac{\mathscr{M}(\mathscr{I}_L(u))}{s}\right)^{\mathfrak{p}_-^\star/p_+} \right\} \\
&\le \kappa \max\left\{ \Phi(u)^{\mathfrak{p}_+^\star/p_-}, \Phi(u)^{\mathfrak{p}_-^\star/p_+} \right\},
\end{aligned}
$$

where $\kappa = (p_+)^{\mathfrak{p}_+^\star/p_-}\mathfrak{K}/\min\{s^{\mathfrak{p}_+^\star/p_-}, s^{\mathfrak{p}_-^\star/p_+}\}$. Therefore, taking r so close to zero that $0<|r|<1$ and putting $\mathcal{K} = \min\left\{\kappa^{-p_-/\mathfrak{p}_+^\star}, \kappa^{-p_+/\mathfrak{p}_-^\star}\right\}$, we have by (A.1) and the facts that $u \equiv 0 \in \Psi_2^{-1}(I^r)$ and $\Psi_2(0) = 0$,

$$
\varphi_2(r) \ge \frac{1}{|r|} \inf_{v \in \Psi_2^{-1}(r)} \Phi(v) \ge \mathcal{K} \min\{|r|^{p_-/\mathfrak{p}_+^\star - 1}, |r|^{p_+/\mathfrak{p}_-^\star - 1}\} = \mathcal{K}|r|^{p_+/\mathfrak{p}_-^\star - 1},
$$

being $p_- \le p_+ < \mathfrak{p}_-^\star \le \mathfrak{p}_+^\star$. This implies that $\lim\limits_{r\to 0^-} \varphi_2(r) = \infty$.

In conclusion, also in this setting, (2.30) holds, so that the proof can be continued and ended exactly as in Theorem 2.2. □

Appendix A. Auxiliary results

We start with recalling a slight variant of Theorem 3.4 of [8] given in [3], which is used throughout the paper. Let $(X, \|\cdot\|)$ be a reflexive real Banach space, with (topological) dual space $(X^\star, \|\cdot\|_\star)$. Assume that Φ and Ψ are two functionals on X, verifying the following hypotheses.

- $(\mathcal{H}_1)$ *Φ and Ψ are weakly lower semi–continuous and continuously Gâteaux differentiable in X, and Ψ is nonconstant;*
- $(\mathcal{H}_2)$ *$\Phi' : X \to X^\star$ has the $(\mathscr{S}_+)$ property, i.e. for every sequence $(u_k)_k \subset X$ such that $u_k \rightharpoonup u$ weakly in X and $\limsup\limits_{k\to\infty} \langle \Phi'(u_k), u_k - u\rangle \le 0$, then $u_k \to u$ strongly in X;*
- $(\mathcal{H}_3)$ *$\Psi' : X \to X^\star$ is a compact operator;*
- $(\mathcal{H}_4)$ *there exists an interval $I \subset \mathbb{R}$ such that the one parameter family of functionals $J_\lambda = \Phi + \lambda\Psi$, $\lambda \in I$, is coercive in X, i.e. for all $\lambda \in I$*

$$
\lim_{\|u\|\to\infty} J_\lambda(u) = \infty.
$$

Given $r \in (\inf_{u\in X}\Psi(u), \sup_{u\in X}\Psi(u))$, we introduce the two functions

$$
\begin{aligned}
\varphi_1(r) &= \inf_{u \in \Psi^{-1}(I_r)} \frac{\inf\limits_{v\in\Psi^{-1}(r)} \Phi(v) - \Phi(u)}{\Psi(u) - r}, \quad I_r = (-\infty, r), \\
\varphi_2(r) &= \sup_{u \in \Psi^{-1}(I^r)} \frac{\inf\limits_{v\in\Psi^{-1}(r)} \Phi(v) - \Phi(u)}{\Psi(u) - r}, \quad I^r = (r, \infty).
\end{aligned}
\tag{A.1}
$$

Theorem Appendix A.1 (Theorem 2.1 of [3]). *Under* $(\mathcal{H}_1)$–$(\mathcal{H}_4)$ *and the existence of*

$$r \in \left(\inf_{u \in X} \Psi(u), \sup_{u \in X} \Psi(u) \right) \quad \text{such that} \quad \varphi_1(r) < \varphi_2(r), \tag{A.2}$$

the following properties hold.

(*i*) *The functional* J_λ *admits at least one critical point for every* $\lambda \in I$.
(*ii*) *If furthermore* $(\varphi_1(r), \varphi_2(r)) \cap I \neq \emptyset$, *then*
  (*a*) J_λ *has at least three critical points for every* $\lambda \in (\varphi_1(r), \varphi_2(r)) \cap I$.
  (*b*) $J_{\varphi_1(r)}$ *has at least two critical points, provided that* $\varphi_1(r) \in I$.
  (*c*) $J_{\varphi_2(r)}$ *has at least two critical points, provided that* $\varphi_2(r) \in I$.

Throughout the paper $L^\sigma(\Omega)$, $\sigma \geq 1$, *denotes the standard Lebesgue space, endowed with the canonical norm* $\|\cdot\|_\sigma$, *and* σ' *is the conjugate exponent of* σ. Moreover, if ω is a weight on Ω and $\sigma \in [1, \infty)$, then $L^\sigma(\Omega, \omega)$, $\sigma \geq 1$, is the weighted Lebesgue space equipped with the norm

$$\|u\|_{\sigma,\omega} = \left(\int_\Omega \omega(x) |u(x)|^\sigma dx \right)^{1/\sigma}.$$

The Cartesian product $[L^\sigma(\Omega, \omega)]^m$, $m \geq 1$, is endowed with the norm

$$\|\varphi\|_{\sigma,\omega} = \left(\int_\Omega \omega(x) |\varphi(x)|_m^\sigma dx \right)^{1/\sigma},$$

where $|\cdot|_m$ is the m–Euclidean norm on $\mathbb{R}^m$ and $\varphi = (\varphi_1, \dots, \varphi_m)$.

Let us now state a useful result for general vector–valued weighted Lebesgue spaces, which is well–known in the framework of the standard Lebesgue spaces. The proof is left to the reader, see also [39] for $m = 1$.

Lemma Appendix A.1. *If* $(\varphi_k)_k$ *and* φ *are in* $[L^\sigma(\Omega, \omega)]^m$ *and* $\varphi_k \to \varphi$ *in* $[L^\sigma(\Omega, \omega)]^m$ *as* $k \to \infty$, *then there exist a subsequence* $(\varphi_{k_j})_j$ *of* $(\varphi_k)_k$ *and a function* $h \in L^\sigma(\Omega, \omega)$ *such that a.e. in* Ω

$$(i) \quad \varphi_{k_j} \to \varphi \text{ as } j \to \infty; \qquad (ii) \quad |\varphi_{k_j}| \leq h \quad \text{for all } j \in \mathbb{N}.$$

We now turn to the more concrete Banach spaces used in the study of (1.1) and present some simple results which do not seem to be so well known.

Lemma Appendix A.2. *Assume that* $f : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$, $f = f(x, v) \not\equiv 0$, *is a Carathéodory function, satisfying* $(\mathcal{F})$–(*a*) *of Section* 3.1. *The Nemytskii operators* $\mathcal{N}_p : [L^p(\Omega, w)]^d \to [L^{p'}(\Omega, w)]^d$ *and* $\mathcal{N}_f : [L^q(\Omega, w)]^d \to [L^{q'}(\Omega, w^{1/(1-q)})]^d$, *defined by* $\mathcal{N}_p(u) = |u|^{p-2}u$ *and* $\mathcal{N}_f(u) = f(\cdot, u(\cdot))$ *respectively, are continuous.*

Proof. Let $(u_k)_k \subset [L^p(\Omega, w)]^d$ be such that $u_k \to u$ in $[L^p(\Omega, w)]^d$ as $k \to \infty$. We prove that $\mathcal{N}_p(u_k) \to \mathcal{N}_p(u)$ in $[L^{p'}(\Omega, w)]^d$ as $k \to \infty$.

Fix a subsequence $(u_{k_j})_j$ of $(u_k)_k$. By Lemma Appendix A.1, with $m = d$, $\sigma = p$ and $\omega = w$, there exist a subsequence, still denoted by $(u_{k_j})_j$, and a function h in $L^p(\Omega, w)$ satisfying $\mathcal{N}_p(u_{k_j}) \to \mathcal{N}_p(u)$ a.e. in Ω and $|\mathcal{N}_p(u_{k_j})| \le h^{p-1} \in L^{p'}(\Omega, w)$. Hence, $w|\mathcal{N}_p(u_{k_j}) - \mathcal{N}_p(u)|^{p'} \le 2^{p'} wh^p \in L^1(\Omega)$. Now, by the dominated convergence theorem, $\mathcal{N}_p(u_{k_j}) \to \mathcal{N}_p(u)$ in $[L^{p'}(\Omega, w)]^d$. Therefore, the entire sequence $(\mathcal{N}_p(u_k))_k$ converges to $\mathcal{N}_p(u)$ in $[L^{p'}(\Omega, w)]^d$ as $k \to \infty$.

Similarly, let $(u_k)_k \subset [L^q(\Omega, w)]^d$ be such that $u_k \to u$ in $[L^q(\Omega, w)]^d$ as $k \to \infty$. We assert that $\mathcal{N}_f(u_k) \to \mathcal{N}_f(u)$ in $[L^{q'}(\Omega, w^{1/(1-q)})]^d$ as $k \to \infty$. Indeed, fix a subsequence $(u_{k_j})_j$ of $(u_k)_k$. There exist a subsequence, still denoted by $(u_{k_j})_j$, and a function $h \in L^q(\Omega, w)$ satisfying (i) and (ii) of Lemma Appendix A.1, with $m = d$, $\sigma = q$ and $\omega = w$, that is $u_{k_j} \to u$ a.e. in Ω and $|u_{k_j}| \le h$ a.e. in Ω for all $j \in \mathbb{N}$. In particular, $|\mathcal{N}_f(u_{k_j}) - \mathcal{N}_f(u)|^{q'} w^{1/(1-q)} \to 0$ a.e. in Ω, being $f(x, \cdot)$ continuous for a.a. $x \in \Omega$. Furthermore, $|\mathcal{N}_f(u_{k_j}) - \mathcal{N}_f(u)|^{q'} w^{1/(1-q)} \le \kappa w(1 + h^q) \in L^1(\Omega)$, $\kappa = (2C_f)^{q'} 2^{q'-1}$, by $(\mathcal{F})$–(a), being $w \in L^{\varpi}(\Omega) \subset L^1(\Omega)$, since $\varpi > 1$ and Ω is bounded. Hence, by the dominated convergence theorem $\mathcal{N}_f(u_{k_j}) \to \mathcal{N}_f(u)$ in $[L^{q'}(\Omega, w^{1/(1-q)})]^d$. Therefore the entire sequence $\big(\mathcal{N}_f(u_k)\big)_k$ converges to $\mathcal{N}_f(u)$ in $[L^{q'}(\Omega, w^{1/(1-q)})]^d$ as $k \to \infty$, as asserted. $\square$

We end the section with two useful results, which seem not to be proven before.

Proposition Appendix A.1. *The space* $\left([W_0^{L,p}(\Omega)]^d, \|\cdot\|_{d,L,p}\right)$ *is uniformly convex.*

Proof. The vector–valued space $\left([W_0^{L,p}(\Omega)]^d, \|\cdot\|_{d,L,p}\right)$ is the Cartesian product of d copies of the scalar space $W_0^{L,p}(\Omega)$ endowed with the norm $\|u\|_{L,p}$ defined in (2.1). It is enough to prove that $\left(W_0^{L,p}(\Omega), \|\cdot\|_{L,p}\right)$ is uniformly convex. Indeed, this implies that $([W_0^{L,p}(\Omega)]^d, \|\cdot\|_{d,L,p})$ is uniformly convex, by Theorem 1.22 of [36]. We distinguish two cases depending on whether L is even or odd.

Case $L = 2j$, $j = 1, 2, \dots$. Fix $\varepsilon \in (0, 2)$ and let u, $v \in W_0^{2j,p}(\Omega)$ be such that $\|u\|_{L,p} = \|v\|_{L,p} = 1$ and $\|u - v\|_{L,p} \ge \varepsilon$.

Consider first the case $p \in [2, \infty)$. By (35) of Lemma 2.27 of [36], we have that for all z, $\zeta \in \mathbb{R}$

$$\left|\frac{z+\zeta}{2}\right|^p + \left|\frac{z-\zeta}{2}\right|^p \le \frac{1}{2}(|z|^p + |\zeta|^p).$$

Hence,

$$\begin{aligned}\left\|\frac{u+v}{2}\right\|_{L,p}^p + \left\|\frac{u-v}{2}\right\|_{L,p}^p &= \int_\Omega \left(\left|\frac{\Delta^j u + \Delta^j v}{2}\right|^p + \left|\frac{\Delta^j u - \Delta^j v}{2}\right|^p\right) dx \\ &\le \frac{1}{2}\int_\Omega \left(|\Delta^j u|^p + |\Delta^j v|^p\right) dx = \frac{1}{2}\left(\|u\|_{L,p}^p + \|v\|_{L,p}^p\right) = 1.\end{aligned}$$

This implies that

$$\left\|\frac{u+v}{2}\right\|_{L,p}^{p} \leq 1-\left(\frac{\varepsilon}{2}\right)^{p}$$

and so, taking $\delta = \delta(\varepsilon)$ such that $1-(\varepsilon/2)^p = (1-\delta)^p$, the proof of this case is concluded.

If $p \in (1,2)$, then by Theorem 2.7 of [36]

$$\left\|\left|\mathcal{D}_L\left(\frac{u+v}{2}\right)\right|^{p'}\right\|_{p-1}+\left\|\left|\mathcal{D}_L\left(\frac{u-v}{2}\right)\right|^{p'}\right\|_{p-1} \leq \left\|\left|\mathcal{D}_L\left(\frac{u+v}{2}\right)\right|^{p'}+\left|\mathcal{D}_L\left(\frac{u-v}{2}\right)\right|^{p'}\right\|_{p-1}.$$

In other words

$$\left\|\frac{u+v}{2}\right\|_{L,p}^{p'}+\left\|\frac{u-v}{2}\right\|_{L,p}^{p'} \leq \left\|\left|\mathcal{D}_L\left(\frac{u+v}{2}\right)\right|^{p'}+\left|\mathcal{D}_L\left(\frac{u-v}{2}\right)\right|^{p'}\right\|_{p-1}, \tag{A.3}$$

since $|\Delta^j\psi|^{p'} \in L^{p-1}(\Omega)$ and $\||\mathcal{D}_L\psi|^{p'}\|_{p-1} = \|\mathcal{D}_L\psi\|_p^{p'}$ for all $\psi \in W_0^{L,p}(\Omega)$. Moreover, being $1<p<2$, by (34) of Lemma 2.27 of [36]

$$\left|\frac{z+\zeta}{2}\right|^{p'}+\left|\frac{z-\zeta}{2}\right|^{p'} \leq \left[\frac{1}{2}(|z|^p+|\zeta|^p)\right]^{1/(p-1)}$$

for all z, $\zeta \in \mathbb{R}$. Hence,

$$\begin{aligned}\left\|\left|\mathcal{D}_L\left(\frac{u+v}{2}\right)\right|^{p'}+\left|\mathcal{D}_L\left(\frac{u-v}{2}\right)\right|^{p'}\right\|_{p-1} &\leq \left[\int_\Omega \frac{1}{2}\left(|\Delta^j u|^p+|\Delta^j v|^p\right)dx\right]^{1/(p-1)} \\ &= \left(\frac{1}{2}\|u\|_{L,p}^p+\frac{1}{2}\|v\|_{L,p}^p\right)^{1/(p-1)} = 1.\end{aligned} \tag{A.4}$$

Combining together (A.3) and (A.4), we get

$$\left\|\frac{u+v}{2}\right\|_{L,p}^{p'} \leq 1-\left\|\frac{u-v}{2}\right\|_{L,p}^{p'} \leq 1-\left(\frac{\varepsilon}{2}\right)^{p'}.$$

It is enough to take $\delta = \delta(\varepsilon)$ such that $1-(\varepsilon/2)^{p'} = (1-\delta)^{p'}$ in order to conclude the proof also in the case $1<p<2$.

Case $L = 2j-1$, $j = 1,2,\ldots$. Consider the vector–valued space $[L^p(\Omega)]^n = ([L^p(\Omega)]^n, \|\cdot\|_{[L^p(\Omega)]^n})$, where

$$\|g\|_{[L^p(\Omega)]^n} = \left(\sum_{i=1}^n \|g_i\|_p^p\right)^{1/p} \qquad \text{for all } g=(g_1,\ldots,g_n) \in [L^p(\Omega)]^n.$$

The linear operator $T: W_0^{L,p}(\Omega) \to [L^p(\Omega)]^n$, defined for all $u \in W_0^{L,p}(\Omega)$ by

$$T(u) = \left(\partial_{x_1}\Delta^j u, \ldots, \partial_{x_n}\Delta^j u\right),$$

is isometric. Furthermore, the space $[L^p(\Omega)]^n$ is uniformly convex, by Theorem 3 of [40], since $(L^p(\Omega), \|\cdot\|_p)$ is uniformly convex itself. Hence, also $\left(W_0^{L,p}(\Omega), \|\cdot\|_{L,p}\right)$

is uniformly convex, being isometric to a uniformly convex Banach space. This concludes the proof. □

Proposition Appendix A.2. *The space* $\left([W_0^{L,p}(\Omega)]^d, \|\cdot\|\right)$ *is uniformly convex.*

Proof. Fix $\varepsilon \in (0,2)$ and let $u, v \in [W_0^{L,p}(\Omega)]^d$ be such that $\|u\| = \|v\| = 1$ and $\|u - v\| \geq \varepsilon$.

Consider first the case $p \in [2, \infty)$. By (A.2) of Lemma A.1 of [16], we have that for all z, $\zeta \in \mathbb{R}^N$

$$\left|\frac{z+\zeta}{2}\right|_N^p + \left|\frac{z-\zeta}{2}\right|_N^p \leq \frac{1}{2}(|z|_N^p + |\zeta|_N^p).$$

Hence, with $z = \mathcal{D}_L u$, $\zeta = \mathcal{D}_L v \in \mathbb{R}^N$, we get

$$\begin{aligned}\left\|\frac{u+v}{2}\right\|_p^p + \left\|\frac{u-v}{2}\right\|_p^p &= \int_\Omega \left(\left|\frac{\mathcal{D}_L u + \mathcal{D}_L v}{2}\right|_N^p + \left|\frac{\mathcal{D}_L u - \mathcal{D}_L v}{2}\right|_N^p\right) dx \\ &\leq \frac{1}{2}\int_\Omega (|\mathcal{D}_L u|_N^p + |\mathcal{D}_L v|_N^p)\, dx = \frac{1}{2}\left(\|u\|^p + \|v\|^p\right) = 1.\end{aligned}$$

This implies that

$$\left\|\frac{u+v}{2}\right\|^p \leq 1 - \left(\frac{\varepsilon}{2}\right)^p.$$

It is enough to take $\delta = \delta(\varepsilon)$ such that $1 - (\varepsilon/2)^p = (1-\delta)^p$, in order to conclude the proof.

If $p \in (1,2)$, then by Theorem 2.7 of [36]

$$\begin{aligned}&\left\|\left|\mathcal{D}_L\left(\frac{u+v}{2}\right)\right|_N^{p'}\right\|_{p-1} + \left\|\left|\mathcal{D}_L\left(\frac{u-v}{2}\right)\right|_N^{p'}\right\|_{p-1} \\ &\qquad\leq \left\|\left|\mathcal{D}_L\left(\frac{u+v}{2}\right)\right|_N^{p'} + \left|\mathcal{D}_L\left(\frac{u-v}{2}\right)\right|_N^{p'}\right\|_{p-1}.\end{aligned}$$

In other words

$$\left\|\frac{u+v}{2}\right\|^{p'} + \left\|\frac{u-v}{2}\right\|^{p'} \leq \left\|\left|\mathcal{D}_L\left(\frac{u+v}{2}\right)\right|_N^{p'} + \left|\mathcal{D}_L\left(\frac{u-v}{2}\right)\right|_N^{p'}\right\|_{p-1}, \tag{A.5}$$

since $|\mathcal{D}_L\phi|_N^{p'} \in L^{p-1}(\Omega)$ and $\||\mathcal{D}_L\phi|_N^{p'}\|_{p-1} = \||\mathcal{D}_L\phi|_N\|_p^{p'}$ for all $\phi \in [W_0^{L,p}(\Omega)]^d$. Moreover, being $1 < p < 2$, by (A.1) of Lemma A.1 of [16], we have that for all z, $\zeta \in \mathbb{R}^N$

$$\left|\frac{z+\zeta}{2}\right|_N^{p'} + \left|\frac{z-\zeta}{2}\right|_N^{p'} \leq \left(\frac{1}{2}|z|_N^p + \frac{1}{2}|\zeta|_N^p\right)^{1/(p-1)}.$$

Hence, with $z = \mathcal{D}_L u$, $\zeta = \mathcal{D}_L v \in \mathbb{R}^N$, we get

$$\begin{aligned}\left\| \left| \mathcal{D}_L \left(\frac{u+v}{2} \right) \right|_N^{p'} + \left| \mathcal{D}_L \left(\frac{u-v}{2} \right) \right|_N^{p'} \right\|_{p-1} &\leq \left[\frac{1}{2} \int_\Omega (|\mathcal{D}_L u|_N^p + |\mathcal{D}_L v|_N^p)\, dx \right]^{1/(p-1)} \\ &= \left(\frac{1}{2}\|u\|^p + \frac{1}{2}\|v\|^p \right)^{1/(p-1)} = 1.\end{aligned} \tag{A.6}$$

Combining together (A.5) with (A.6), we obtain

$$\left\| \frac{u+v}{2} \right\|^{p'} \leq 1 - \left\| \frac{u-v}{2} \right\|^{p'} \leq 1 - \left(\frac{\varepsilon}{2} \right)^{p'}.$$

It is enough to take $\delta = \delta(\varepsilon)$ such that $1 - (\varepsilon/2)^{p'} = (1-\delta)^{p'}$ in order to conclude the proof. □

Appendix B. Supplementary results for (1.6)

The weighted variable exponent Lebesgue space $L^{\sigma(\cdot)}(\Omega, \omega)$ defined in Section 3 is a Banach space. First, $\|u\|_{\sigma(\cdot),\omega} = 0$ if and only if $u \equiv 0$ a.e. in Ω and $\|\lambda u\|_{\sigma(\cdot),\omega} = |\lambda| \cdot \|u\|_{\sigma(\cdot),\omega}$ for all $u \in L^{\sigma(\cdot)}(\Omega, \omega)$ and $\lambda \in \mathbb{R}$. Moreover, fixed u, $v \in L^{\sigma(\cdot)}(\Omega, \omega)$ it is clear that $\|u+v\|_{\sigma(\cdot),\omega} = \|u\|_{\sigma(\cdot),\omega} + \|v\|_{\sigma(\cdot),\omega}$, whenever either $u = 0$ or $v = 0$. Hence, let us assume that $\|u\|_{\sigma(\cdot),\omega} > 0$ and $\|v\|_{\sigma(\cdot),\omega} > 0$. Take $s > \|u\|_{\sigma(\cdot),\omega} = \|\omega^{1/\sigma} u\|_{\sigma(\cdot)}$ and $t > \|v\|_{\sigma(\cdot),\omega} = \|\omega^{1/\sigma} v\|_{\sigma(\cdot)}$. Then, $\|\omega^{1/\sigma} u/s\|_{\sigma(\cdot)} < 1$ and $\|\omega^{1/\sigma} v/t\|_{\sigma(\cdot)} < 1$, so that $\rho_{\sigma(\cdot)} \left(\omega^{1/\sigma} u/s\right) < 1$ and $\rho_{\sigma(\cdot)} \left(\omega^{1/\sigma} v/t\right) < 1$ by (3.2). Therefore,

$$\rho_{\sigma(\cdot)} \left(\frac{\omega^{1/\sigma}(u+v)}{s+t} \right) \leq \frac{s}{s+t} \rho_{\sigma(\cdot)} \left(\frac{\omega^{1/\sigma} u}{s} \right) + \frac{t}{s+t} \rho_{\sigma(\cdot)} \left(\frac{\omega^{1/\sigma} v}{t} \right) < \frac{s}{s+t} + \frac{t}{s+t} = 1.$$

In other words, $\|u+v\|_{\sigma(\cdot),\omega} = \|\omega^{1/\sigma}(u+v)\|_{\sigma(\cdot)} \leq s+t$. In conclusion, in all cases

$$\|u+v\|_{\sigma(\cdot),\omega} \leq \|u\|_{\sigma(\cdot),\omega} + \|v\|_{\sigma(\cdot),\omega}.$$

Hence also $(L^{\sigma(\cdot)}(\Omega, \omega), \|\cdot\|_{\sigma(\cdot),\omega})$ is a normed space. Moreover, $L^{\sigma(\cdot)}(\Omega, \omega)$ inherits all the properties of $L^{\sigma(\cdot)}(\Omega)$ by (3.3).

As in Section 3, the vector–valued weighted variable exponent Lebesgue space $[L^{\sigma(\cdot)}(\Omega, \omega)]^m$ is endowed with the norm

$$\|\varphi\|_{m,\sigma(\cdot),\omega} = \|\, |\varphi|_m \|_{\sigma(\cdot),\omega}.$$

The next lemma is the analogous of Lemma Appendix A.1. For completeness we present the proof.

Lemma Appendix B.1. *Let φ, $(\varphi_k)_k \subset [L^{\sigma(\cdot)}(\Omega, \omega)]^m$ be such that $\varphi_k \to \varphi$ in $[L^{\sigma(\cdot)}(\Omega, \omega)]^m$ as $k \to \infty$. Then there exist a subsequence $(\varphi_{k_j})_j \subset (\varphi_k)_k$ and a function $h \in L^{\sigma(\cdot)}(\Omega, \omega)$ such that for a.a. $x \in \Omega$*

$$(i)\ \varphi_{k_j}(x) \to \varphi(x) \quad \text{as } j \to \infty, \qquad (ii)\ |\varphi_{k_j}(x)| \leq h(x) \quad \text{for all } j \in \mathbb{N}.$$

Proof. Let φ, $(\varphi_k)_k \subset [L^{\sigma(\cdot)}(\Omega,\omega)]^m$ be as in the statement. Clearly $(\varphi_k)_k$ is a Cauchy sequence in $[L^{\sigma(\cdot)}(\Omega,\omega)]^m$ and so there exists a subsequence $(\varphi_{k_j})_j \subset (\varphi_k)_k$ such that $\|\varphi_{k_{j+1}} - \varphi_{k_j}\|_{m,\sigma(\cdot),\omega} \le 2^{-j}$ for all $j \ge 1$. The function

$$g_\ell(x) = \sum_{j=1}^{\ell} |\varphi_{k_{j+1}}(x) - \varphi_{k_j}(x)|_m \quad \text{for a.a. } x \in \Omega,$$

is non–negative in $L^{\sigma(\cdot)}(\Omega,\omega)$ and $(g_\ell)_\ell$ is non–decreasing, with $\|g_\ell\|_{\sigma(\cdot),\omega} \le \sum_{j=1}^{\ell} \|\varphi_{k_{j+1}} - \varphi_{k_j}\|_{m,\sigma(\cdot),\omega} \le 1$. Hence, by the monotone convergence theorem given in Lemma 3.2.8–(b) of [19] and (3.3), the sequence $(g_\ell)_\ell$ converges a.e. to some g and $g \in L^{\sigma(\cdot)}(\Omega,\omega)$ by the Fatou Lemma 3.2.8–(a) of [19] and by (3.3). Hence g is finite a.e. in Ω. For all $\ell > l \ge 2$ and a.a. $x \in \Omega$

$$\begin{aligned}|\varphi_{k_\ell}(x) - \varphi_{k_l}(x)|_m &\le |\varphi_{k_\ell}(x) - \varphi_{k_{\ell-1}}(x)|_m + \ldots + |\varphi_{k_{l+1}}(x) - \varphi_{k_l}(x)|_m \\ &= g_{\ell-1}(x) - g_{l-1}(x) \le g(x) - g_{l-1}(x).\end{aligned}$$

Thus the sequence $(\varphi_{k_j}(x))_j$ is a Cauchy sequence in $\mathbb{R}^m$ and it converges to some $\varphi(x) \in \mathbb{R}^m$ for a.a. $x \in \Omega$. Now, for all $j \ge 2$

$$|\varphi(x) - \varphi_{k_j}(x)|_m \le g(x), \tag{B.1}$$

so that $\varphi \in [L^{\sigma(\cdot)}(\Omega,\omega)]^m$. Therefore, $\varphi_{k_j} \to \varphi$ in $[L^{\sigma(\cdot)}(\Omega,\omega)]^m$ as $j \to \infty$ by the dominated convergence theorem, given in Lemma 3.2.8–(c) of [19] and (3.3), where $E^{\sigma(\cdot)}(\Omega,\omega) = L^{\sigma(\cdot)}(\Omega,\omega)$, being $\sigma_+ < \infty$. Finally, $|\varphi_{k_j}|_m \le |\varphi|_m + g \in L^{\sigma(\cdot)}(\Omega,\omega)$ a.e. in Ω for all $j \ge 2$ by (B.1). Hence, it is enough to choose $h = |\varphi|_m + g$ and the proof is complete. □

Lemma Appendix B.2. *Assume that $f : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$, $f = f(x,v) \not\equiv 0$, is a Carathéodory function, satisfying $(\mathcal{F})$–$(a)'$ of Section 3. The Nemytskii operator $\mathcal{N}_f(u) = f(\cdot,u(\cdot))$, $\mathcal{N}_f : [L^{q(\cdot)}(\Omega,w)]^d \to [L^{q'(\cdot)}(\Omega,w^{1/(1-q)})]^d$, is continuous.*

Proof. Let $(u_k)_k \subset [L^{q(\cdot)}(\Omega,w)]^d$ be such that $u_k \to u$ in $[L^{q(\cdot)}(\Omega,w)]^d$ as $k \to \infty$. We assert that $\mathcal{N}_f(u_k) \to \mathcal{N}_f(u)$ in $[L^{q'(\cdot)}(\Omega,w^{1/(1-q)})]^d$ as $k \to \infty$. Indeed, fix a subsequence $(u_{k_j})_j$ of $(u_k)_k$. There exist a subsequence, still denoted by $(u_{k_j})_j$, and a function $h \in L^{q(\cdot)}(\Omega,w)$ satisfying (i) and (ii) of Lemma Appendix B.1, with $m = d$, $\sigma = q$ and $\omega = w$, that is $u_{k_j} \to u$ a.e. in Ω and $|u_{k_j}| \le h$ a.e. in Ω for all $j \in \mathbb{N}$. In particular, $|\mathcal{N}_f(u_{k_j}) - \mathcal{N}_f(u)|^{q'(x)} w^{1/(1-q(x))} \to 0$ a.e. in Ω, being $f(x,\cdot)$ continuous for a.a. $x \in \Omega$. Furthermore, by $(\mathcal{F})$–$(a)'$,

$$|\mathcal{N}_f(u_{k_j}) - \mathcal{N}_f(u)|^{q'(x)} w^{1/(1-q(x))} \le \kappa w(1 + h^{q(x)}) \in L^1(\Omega),$$

where $\kappa = 2^{2q'_+-1}\max\{C_f^{q'_+}, C_f^{q'_-}\}$, since $w \in L^{\varpi}(\Omega) \subset L^1(\Omega)$, being $\varpi > 1$ and Ω bounded by (3.8). Hence, $\mathcal{N}_f(u_{k_j}) \to \mathcal{N}_f(u)$ in $[L^{q'(\cdot)}(\Omega,w^{1/(1-q)})]^d$ by the classical dominated convergence theorem. Therefore, the entire sequence $(\mathcal{N}_f(u_k))_k$ converges to $\mathcal{N}_f(u)$ in $[L^{q'(\cdot)}(\Omega,w^{1/(1-q)})]^d$ as $k \to \infty$, as requested. □

Proposition Appendix B.1. *The space* $\left([W_0^{L,p(\cdot)}(\Omega)]^d, \|\cdot\|\right)$ *is uniformly convex.*

Proof. By Theorem 2.4.14 of [19] it is enough to show that $\rho(u) = \rho_{p(\cdot)}(|\mathcal{D}_L(u)|_N)$ is uniformly convex, since property Δ_2 holds, with $K = 2^{p_+} > 2$, being $1 < p_+ < \infty$. By Theorem 2.4.11 of [19], the uniform convexity of $\rho_{p(\cdot)}(|\cdot|_N)$ follows by proving that $\varphi(x, u) = |u|_N^{p(x)}$ is uniformly convex in $\mathbb{R}^N$ for all $x \in \Omega$. Since $p_- > 1$ it is enough to show that $\varphi(u) = |u|_N^{p_-}$ is uniformly convex in $\mathbb{R}^N$. Indeed, if $\varphi(u) = |u|_N^{p_-}$ is uniformly convex in $\mathbb{R}^N$, then by Definition 2.4.5 of [19] for all $\varepsilon > 0$ there is $\delta = \delta(\varepsilon, p_-) \in (0, 1)$ such that for all $u, v \in \mathbb{R}^N$, with $|u-v|_N > \varepsilon \max\{|u|_N, |v|_N\}$, we have

$$\left|\frac{u+v}{2}\right|_N^{p_-} \le (1-\delta)\frac{|u|_N^{p_-} + |v|_N^{p_-}}{2}.$$

Therefore, for all $x \in \Omega$

$$\left|\frac{u+v}{2}\right|_N^{p(x)} \le (1-\delta)^{p(x)/p_-} \left(\frac{|u|_N^{p_-} + |v|_N^{p_-}}{2}\right)^{p(x)/p_-} \le (1-\delta)\frac{|u|_N^{p(x)} + |v|_N^{p(x)}}{2},$$

as required.

To complete the proof it remains to show that $\varphi(u) = |u|_N^{p_-}$ is uniformly convex in $\mathbb{R}^N$. To this aim we fix $\varepsilon \in (0, 2)$ and $u, v \in \mathbb{R}^N$, with $|u - v|_N > \varepsilon \max\{|u|_N, |v|_N\}$. If $\big| |u|_N - |v|_N \big| > \varepsilon \max\{|u|_N, |v|_N\}/2$, then there exists $\delta = \delta(\varepsilon/2) \in (0, 1)$ such that

$$\left(\frac{|u|_N + |v|_N}{2}\right)^{p_-} \le (1-\delta)\frac{|u|_N^{p_-} + |v|_N^{p_-}}{2},$$

since $\tau \mapsto \tau^{p_-}$ is uniformly convex in $\mathbb{R}_0^+$, as proved in the Remark 2.4.6 of [19], being $1 < p_- < \infty$. Therefore the claim follows at once since $|u+v|_N \le |u|_N + |v|_N$. Let us then consider the case when $2\big| |u|_N - |v|_N \big| \le \varepsilon \max\{|u|_N, |v|_N\}$. Hence,

$$|u - v|_N > \varepsilon \max\{|u|_N, |v|_N\} \ge 2\big| |u|_N - |v|_N \big|.$$

By the parallelogram identity

$$\begin{aligned}\left|\frac{u+v}{2}\right|_N^2 &= \frac{|u|_N^2}{2} + \frac{|v|_N^2}{2} - \left|\frac{u-v}{2}\right|_N^2 \le \frac{|u|_N^2 + |v|_N^2}{2} - \frac{3}{4}\cdot\left|\frac{u-v}{2}\right|_N^2 - \left(\frac{|u|_N - |v|_N}{2}\right)^2 \\ &= \left(\frac{|u|_N + |v|_N}{2}\right)^2 - \frac{3}{16}|u-v|_N^2 \le \left(1 - \frac{3\varepsilon^2}{16}\right)\cdot\left(\frac{|u|_N + |v|_N}{2}\right)^2.\end{aligned}$$

In conclusion, it is enough to take $\delta = 1 - \sqrt{1 - 3\varepsilon^2/16} \in (0, 1)$ in order to get

$$\left|\frac{u+v}{2}\right|_N^{p_-} \le (1-\delta)\left(\frac{|u|_N + |v|_N}{2}\right)^{p_-} \le (1-\delta)\frac{|u|_N^{p_-} + |v|_N^{p_-}}{2}. \qquad \square$$

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