SPHERES AND HOMOLOGY SPHERES

OBTAINED BY SURGERY

M. De Simone and A.I. Telloni

Abstract We study a class of reflexive links and the surgery manifolds arising from them. We determine geometric presentations for the fundamental group and a Rail-Road system for any surgery manifold. Finally we describe the surgery homology spheres as double branched coverings of $\mathbb{S}^3$ and draw explicitely the branch sets, which are prime three-bridge knots.

2000 Mathematics Subject Classification: 57M25; 57M12

Keywords: reflexive links, Dehn surgery, homology sphere, lens space, branched covering, fundamental group.

1 Introduction

Dehn surgery theory and branched coverings over knots and links are different techniques of representation of 3-manifolds. It is known that all 3-manifolds can be constructed by surgery; on the other hand, any closed 3-manifold can be represented as a branched covering of a suitable link in $\mathbb{S}^3$. Many researchers are interested in the connections between these two methods to represent closed 3-manifolds (see, for example [KK], [A]). A celebrated result due to Gordon and Luecke ([GL]) says that non-trivial Dehn surgery on a non-trivial knot never yield the 3-sphere, while there exist links admitting non-trivial surgeries yielding $\mathbb{S}^3$. For example, any $1/n$-surgery on an unknotted component of a link in $\mathbb{S}^3$ gives the 3-sphere again. These links are called *reflexive* and they are studied by many researchers (see [O], [B], [MOS]). In [Te] M. Teragaito constructed a family of infinitely many unsplittable links of n-components in the 3-sphere with a non-trivial surgery yielding $\mathbb{S}^3$. In this paper we introduce a class of reflexive links L_n, $n \geq 2$, with an arbitrary number of unknotted components and investigate the manifolds arising by performing Dehn surgery on them. One of our results is that any link L_n admits infinitely many surgeries yielding the 3-sphere and two infinite classes of lens space surgeries. We determine geometric presentations for the fundamental group of the surgery manifolds arising from L_n and focus on the homology spheres. We recognize them as double branched coverings of the 3-sphere over some classes of different prime knots, for which we give explicit planar projections and 2-generator presentations for the knot groups. As a corollary of our study, we extend the Theorem 2.1 of [Te].

2 The links L_n

Let $L_n = K_1 \cup \cdots \cup K_{n+2}$ the link depicted in Figure 1. We denote $M = L(\frac{p_1}{q_1}, \ldots, \frac{p_{n+2}}{q_{n+2}})$ the manifold obtained by surgery on L_n with surgery coefficient $\frac{p_i}{q_i}$ on the component K_i, for $i = 1, \ldots, n+2$. This means that M is obtained by gluing $n+2$ solid tori to the exterior of the link L_n along their boundaries according to the surgery instructions (for more details see, for example, [R]). Since each surgery manifold is uniquely determined by the slopes $\gamma_i = \frac{p_i}{q_i}$, we assume that the integers p_i and q_i are coprime, and $p_i \geq 0$, for $i = 1, \ldots, n+2$. By using the Wirtinger algorithm on the planar projection in Figure 1, we obtain our first theorem. Recall that, given two generators of a group presentation, the symbol $[x, y]$ denotes the commutator word $xyx^{-1}y^{-1}$.

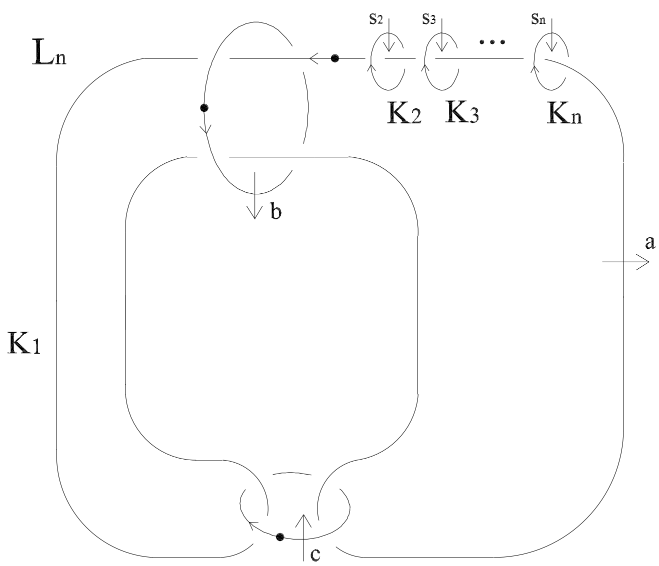

Figure 1. The link L_n

Theorem 1. *The group of L_n, that is, the fundamental group of $\mathbb{S}^3 \setminus L_n$ admits the following presentation*

$$\begin{aligned}\pi(L_n) = \langle a, b, c, s_2, \ldots, s_n \ :& \ [a, s_j] = 1, \ (j = 2, \ldots, n) \\ & bc^{-1}b^{-1}abcb^{-1}c^{-1}a^{-1}c = 1, bcb^{-1}aba^{-1}c^{-1}ab^{-1}a^{-1} = 1\rangle\end{aligned}$$

3 The surgery manifolds

For $i = 1, \ldots, n+2$ denote with $(\mathbf{m_i}, \mathbf{l_i})$ a meridian-longitude pair of the component K_i of L_n, so that $\mathbf{l_i}$ is homologous to zero in the complement of K_i (i.e. $(\mathbf{m_i}, \mathbf{l_i})$ is a so called *preferred frame*). Then we have:

$$\begin{array}{lll} \mathbf{m_1} = a & \mathbf{l_1} = bcb^{-1}c^{-1}s_n^{-1} \ldots s_2^{-1} & \\ \mathbf{m_i} = s_i & \mathbf{l_i} = a^{-1} & (i = 2, \ldots n) \qquad (*) \\ \mathbf{m_{n+1}} = b & \mathbf{l_{n+1}} = c^{-1}a^{-1}ca & \\ \mathbf{m_{n+2}} = c & \mathbf{l_{n+2}} = c^{-1}b^{-1}aba^{-1}c & \end{array}$$

where $[\mathbf{m_i}, \mathbf{l_i}] = 1$, for $i = 1, \ldots, n+2$.

A presentation for the fundamental group of the surgery manifold M can be obtained from $\pi(L_n)$ by adding the surgery relations $\mathbf{m_i}^{p_i}\mathbf{l_i}^{q_i} = 1$, for $i = 1, \ldots n+2$. So, we give the following

Theorem 2. *The fundamental group of $M = L(\frac{p_1}{q_1}, \ldots, \frac{p_{n+2}}{q_{n+2}})$ is presented by*

$$\begin{aligned}\pi_1(M) = \langle a, b, c, s_2, \ldots, s_n \ :& \ [a, s_j] = 1, \ (j = 2, \ldots, n) \\ & bc^{-1}b^{-1}abcb^{-1}c^{-1}a^{-1}c = 1 \\ & bcb^{-1}aba^{-1}c^{-1}ab^{-1}a^{-1} = 1 \\ & a^{p_1}bcb^{-1}c^{-1}(s_n^{-1} \ldots s_2^{-1})^{q_1} = 1 \\ & s_i^{p_i}a^{-q_i} = 1, \qquad (i = 2, \ldots, n) \\ & b^{p_{n+1}}(c^{-1}a^{-1}ca)^{q_{n+1}} = 1, \ c^{p_{n+2}}(b^{-1}aba^{-1})^{q_{n+2}} = 1\rangle\end{aligned}$$

Recall that a *spine* of a closed manifold M is a 2-polyhedron K such that M minus a 3-cell retracts to K.

Theorem 3. *The fundamental group of $M = L(\frac{p_1}{q_1}, \ldots, \frac{p_{n+2}}{q_{n+2}})$ is presented* *by*

$$\pi_1(M) = \langle A_1,\dots,A_n,B,C \ : \begin{array}{l} {A_i}^{p_i} = {A_1}^{q_1},\ i=2,\dots,n \\ {A_1}^{-p_1} = B^{q_{n+1}}C^{q_{n+2}}B^{-q_{n+1}}C^{-q_{n+2}}{A_n}^{-q_n}\dots {A_2}^{-q_2}, \\ B^{-p_{n+1}} = C^{-q_{n+2}}{A_1}^{-q_1}C^{q_{n+2}}{A_1}^{q_1} \\ C^{-p_{n+2}} = B^{-q_{n+1}}{A_1}^{q_1}B^{q_{n+1}}{A_1}^{-q_1}\rangle \end{array}$$

Moreover, this presentation is geometric, that is it corresponds to a spine of the manifold M.

Proof. Following a technique explained in [CST], we modify the group presentation in Theorem 2. Since $(p_i, q_i) = 1$, there exist integers α_i, β_i such that $\alpha_i q_i - \beta_i p_i = 1$, for every $i = 1,\dots,n+2$. Define new words

$$\begin{array}{ll} A_i = \mathbf{m_i}^{\alpha_i}\mathbf{l_i}^{\beta_i}, & i = 1,\dots,n \\ B = \mathbf{m_{n+1}}^{\alpha_{n+1}}\mathbf{l_{n+1}}^{\beta_{n+1}} & \\ C = \mathbf{m_{n+2}}^{\alpha_{n+2}}\mathbf{l_{n+2}}^{\beta_{n+2}} & , \end{array}$$

which gives

$$\begin{aligned} A_i^{q_i} &= (\mathbf{m_i}^{\alpha_i}\mathbf{l_i}^{\beta_i})^{q_i} = \mathbf{m_i}^{\alpha_i q_i}\mathbf{l_i}^{\beta_i q_i} = \mathbf{m_i}^{1+\beta_i p_i}\mathbf{l_i}^{\beta_i q_i} = \mathbf{m_i}(\mathbf{m_i}^{p_i}\mathbf{l_i}^{q_i})^{\beta_i} = \mathbf{m_i} \\ A_i^{-p_i} &= (\mathbf{m_i}^{\alpha_i}\mathbf{l_i}^{\beta_i})^{-p_i} = \mathbf{m_i}^{-\alpha_i p_i}\mathbf{l_i}^{-\beta_i p_i} = \mathbf{m_i}^{-\alpha_i p_i}\mathbf{l_i}^{1-\alpha_i p_i} = \mathbf{l_i}(\mathbf{m_i}^{p_i}\mathbf{l_i}^{q_i})^{-\alpha_i} = \mathbf{l_i} \end{aligned}$$

for $i = 1,\dots,n$ and analogously

$$B^{q_{n+1}} = \mathbf{m_{n+1}}, \quad B^{-p_{n+1}} = \mathbf{l_{n+1}}, \quad C^{q_{n+2}} = \mathbf{m_{n+2}}, \quad C^{-p_{n+2}} = \mathbf{l_{n+2}}.$$

Hence

$$a = {A_1}^{q_1}, \quad s_i = {A_i}^{q_i}\ (i = 2,\dots,n), \quad b = B^{q_{n+1}}, \quad c = C^{q_{n+2}}.$$

Substituting these new generators in the relations (*) gives the presentation of the statement. Furthermore, this presentation is geometric, since it is induced by the Rail-Road-system depicted in Figure 2. Here the hexagons correspond to the $n+2$ generators and the closed curves between the hexagons give rise of the relations (for more details on Rail-Road-System see [OS]). □

As a corollary of Theorem 3, we obtain immediately the following results about Teragaito's links L'_n depicted in Figure 3. In fact, the link L_n is obtained from L'_n by adding two simple curves K_{n+1} and K_{n+2} encircling the twist regions and performing suitable twists around them. Hence, any Dehn surgery on L'_n can be described as a Dehn surgery on L_n.

Corollary 4 (Teragaito [Te]) *The surgery manifold $L'(n-2, 0, 1, \ldots, 1)$ obtained from the link L'_n by Dehn surgery with coefficients $n-2$ on the component K'_1, 0 on the component K'_2 and 1 on the other components of L'_n is the 3-sphere.*

Proof. The link L_n is obtained from L'_n by adding the unknotted components K_{n+1} and K_{n+2} and performing a -1-twist around K_{n+1} and a +1-twist around K_{n+2}. Adding an unknotted component with coefficient ∞ and twisting around a component does not change the surgery manifold, provided that the surgery coefficients are modified according to the Kirby calculus. Then, by the equivalence theorem about surgery descriptions of manifolds, $L'(n-2, 0, 1 \ldots, 1)$ is equivalent to $L(\gamma_1, \ldots, \gamma_{n+2})$, where the coefficients of the first n components remain unchanged, $\gamma_{n+1} = -1$, $\gamma_{n+2} = 1$. By Theorem 3, $L_n(n-2, 0 \ldots, 1, -1, 1)$ has trivial fundamental group, hence it is homeomorphic to $\mathbb{S}^3$. □

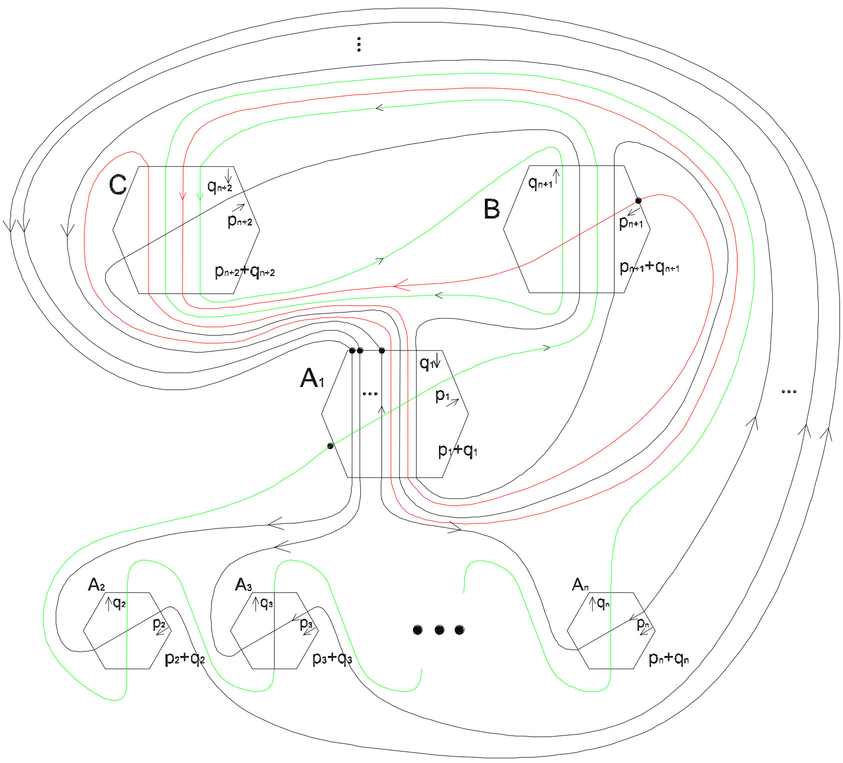

Figure 2. A Rail-Road-system for $M = L(\frac{p_1}{q_1}, \ldots, \frac{p_{n+2}}{q_{n+2}})$

Corollary 5 *The surgery manifold $M' = L'(\frac{p_1}{q_1}, \ldots, \frac{p_n}{q_n})$ with $q_1 = 1$, $p_i = 0$*

for a fixed index i, $|\frac{p_j}{q_j}| = 1$ for $j \neq i$ and $i, j \in \{2, \dots, n\}$ is homeomorphic to $\mathbb{S}^3$.

Proof. As in the previous proof, M' is homeomorphic to $M = L(\frac{p_1}{q_1}, \dots, \frac{p_{n+2}}{q_{n+2}})$ with the fixed values of the first n parameters, $\frac{p_{n+1}}{q_{n+1}} = 1$, $\frac{p_{n+2}}{q_{n+2}} = -1$. By Theorem 3 and the conditions on the surgery coefficients, it is possible to obtain a reduced presentation for $\pi_1(M)$ in which all generators disappear. So $\pi_1(M)$ turns out to be the trivial group, hence $M' \cong M \cong \mathbb{S}^3$. □

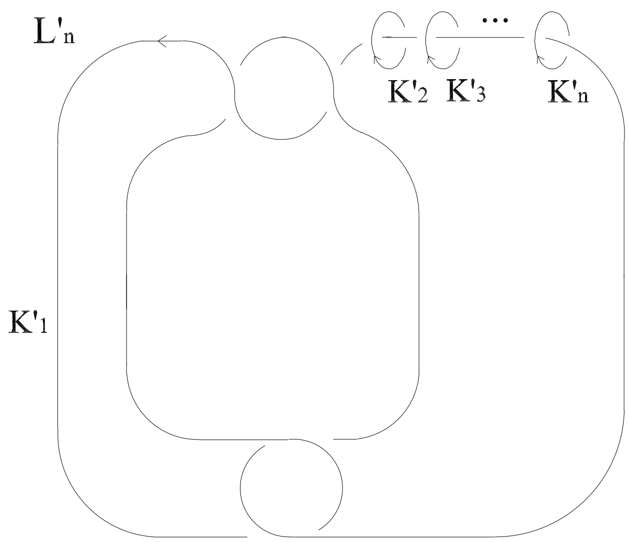

Figure 3. The Teragaito's link L'_n, for $n \geq 2$

4 Surgery homology spheres and lens spaces

Theorem 6. *First homology group of the manifold $M = M(\frac{p_1}{q_1}, \dots, \frac{p_{n+2}}{q_{n+2}})$ is presented by*

$$\begin{aligned}\mathbb{H}_1(M) = \langle A_1, \dots, A_n, B, C \ : \ & [A_i, A_j] = [B, A_i] = [C, A_i] = 1, \ (i, j = 1, \dots, n) \\ & [B, C] = 1, \ {A_1}^{q_1} = {A_i}^{p_i}, \ (i = 2, \dots, n) \\ & {A_1}^{p_1} = {A_2}^{q_2} \dots {A_n}^{q_n}, \ B^{p_{n+1}} = 1, \ C^{p_{n+2}} = 1\rangle,\end{aligned}$$

that is, $\mathbb{H}_1(M) \equiv \mathbb{Z}_{p_{n+1}} \oplus \mathbb{Z}_{p_{n+2}} \oplus G$, where G is the group presented by the generators A_i and the relations of $\mathbb{H}_1(M)$ involving only generators A_i.

Proof. Since $\mathbb{H}_1(M)$ is the abelianization of $\pi_1(M)$, we add the commutators $[x, y] = xyx^{-1}y^{-1}$ between the generators to the presentation of Theorem

3. The statement follows from standard reductions of the obtained group presentation. □

Corollary 7. *The manifold* $M = L(\frac{p_1}{q_1}, \ldots, \frac{p_{n+2}}{q_{n+2}})$ *is a homology sphere if and only if*

$$p_{n+1}p_{n+2}(\prod_{i=1}^{n} p_i - q_1 \sum_{j=2}^{n} P(j)q_j) = \pm 1,$$

where $P(j)$ *denotes the product of all coefficients* p_i, $i \in \{2, \ldots, n\} \setminus \{j\}$.

Proof. The matrix representing the relators of $\mathbb{H}_1(M)$ is

$$\mathcal{H} = \begin{pmatrix} q_1 & -p_2 & 0\ldots & 0 & 0 & 0 & 0 \\ q_1 & 0 & -p_3 & \vdots & 0 & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ q_1 & 0 & 0 & \ldots & -p_n & 0 & 0 \\ 0 & 0 & \ldots & 0 & 0 & p_{n+1} & 0 \\ 0 & 0 & \ldots & 0 & 0 & 0 & p_{n+2} \\ p_1 & -q_2 & -q_3 & \ldots & -q_n & 0 & 0 \end{pmatrix}$$

It is known that M is a homology sphere if and only if $det\mathcal{H} = \pm 1$, and the statement follows by applying successively Laplace theorem to the $(n+2)$-th, $(n+1)$-th and n-th columns. □

Corollary 8. *Let* $p_i, q_i \in \mathbb{Z}$, $(p_i, q_i) = 1$, $p_i \geq 0$, *for* $i = 1, \ldots, n+2$ *and the following conditions*

(a) $p_{n+1} = 1$ *or* $p_{n+2} = 1$

(b.1) $p_i = |p_1 - q_1(\sum_{j=2}^{n} q_j)| = 1$, *for* $i = 2, \ldots, n$

(b.2) $|q_1| = 1$, $p_i = 0$, $|q_i| = 1$ *for a fixed index* $i \in \{2, \ldots, n\}$, $p_j = 1$, *for* $j = 2, \ldots, n$ *and* $j \neq i$

(c) $p_i = 1$, $i = 2, \ldots, n+2$.

If either one of **b.1** *and* **b.2** *holds together with* **a**, *or* **c** *holds, then the manifold* $M = L(\frac{p_1}{q_1}, \ldots, \frac{p_{n+2}}{q_{n+2}})$ *is a homology lens space. In particular, if either* **b.2** *and* **a** *hold or* $q_{n+1}q_{n+2} = 0$ *and* **c** *hold, then* M *is just a lens space.*

Proof. By Theorem 6, the surgery manifold M is a homology lens space if and only if exactly two of the sets $\mathbb{Z}_{p_{n+1}}$, $\mathbb{Z}_{p_{n+2}}$ and G are trivial and the remaining one has finite order. Now, condition **(a)** assures at least one of

$\mathbb{Z}_{p_{n+1}}, \mathbb{Z}_{p_{n+2}}$ is trivial; each one of the conditions (**b.1**) and (**b.2**) gets $G \cong 0$ and condition **(c)** gives $\mathbb{H}_1(M) \cong \mathbb{Z}_{p_1 - q_1(q_2+\cdots+q_n)}$. This completes the proof for the homology lens spaces. To prove the second part, first suppose **b.2** holds and $p_{n+1} = 1$ (the proof is analogous if $p_{n+2} = 1$). The hypotheses allow us to eliminate the generators A_j, for $j \neq i$, $j \in \{1, \ldots, n\}$, and B by using relations $A_j = 1$ and $B = C^{q_{n+2}} C^{-q_{n+2}}$. So we get $\pi_1(M) \equiv \langle C \ : \ C^{p_{n+2}} = 1 \rangle \cong \mathbb{Z}_{p_{n+2}}$. If **(c)** holds and $q_{n+1} = 0$, we can reduce the presentation of Theorem 3 to the equivalent one $\langle A_1^{-p_1+q_1(q_2+\cdots+q_n)} = C^{q_{n+2}} C^{-q_{n+2}}, B = C = 1 \rangle \cong \mathbb{Z}_t$, where $t = -p_1 + q_1(q_2 + \cdots + q_n)$. □

5 Covering properties

Let us denote with $H(q, l, m, r)$ the homology sphere $L_n(\frac{p}{q}, \frac{1}{q_2} \ldots, \frac{1}{q_n}, \frac{1}{l}, \frac{1}{m})$, where $r = p - q \sum_{i=2}^{n} q_i$. By Theorem 4, we have eight classes of surgery homology spheres, depending on the values of l, m, n, $r \in \{-1, 1\}$, for which we determine geometric presentations for the fundamental group and covering properties.

Theorem 9. *The surgery manifold $H(q, l, m, r)$ has Heegaard genus equal to 2 and it is homeomorphic to the 2-fold cyclic covering of $\mathbb{S}^3$ branched over the three-bridge knot $K(q, l, m, n)$.*

Proof. Let us prove the statement for the manifold $H(q, 1, 1, 1)$; the proofs for the other homology spheres are analogous. By Theorem 3, we get a presentation for the fundamental group $\pi(q, 1, 1, 1,)$ of $H(q, 1, 1, 1)$. The special values of the surgery coefficients allow to eliminate the generators C and A_i from the relations $C = {A_1}^q B^{-1} {A_1}^{-q} B$ and $A_i = {A_1}^q$, $i = 2, \ldots, n$. So we get the following presentation for $\pi(q, 1, 1, 1)$

$$\begin{aligned} \langle A_1, B : &{A_1}^{-p+q\sum_{i=2}^{n} q_i} = B{A_1}^q B^{-1} {A_1}^{-q} B B^{-1} ({A_1}^q B^{-1} {A_1}^{-q} B)^{-1} \\ &B^{-1} = ({A_1}^q B^{-1} {A_1}^{-q} B)^{-1} {A_1}^{-q} {A_1}^q B^{-1} {A_1}^{-q} B {A_1}^q \rangle, \end{aligned}$$

which is equivalent to

$$\langle A, B : \ BA^{1-q} BA^q B^{-1} A^{-q} B^{-1} A^q = 1, \ A^{2q} BA^{-q} B^{-1} A^{-q} B = 1 \rangle.$$

This presentation is geometric, that is it corresponds to a spine of the manifold $H(q, 1, 1, 1)$, as stated in Theorem 3, and it is induced by the Heegard diagram $\mathcal{G}(q, 1, 1, 1)$ in Figure 4. In fact, the holes of the diagram correspond to the generators, and the closed curves between the holes correspond to the relators of $\pi(q,1,1,1)$. Hence, the Heegaard genus of $H(q,1,1,1)$ is 2. Moreover, the diagram $\mathcal{G}(1,1,1,1)$ is 2-symmetric, that is it admits two different symmetries of order two. Following a costruction explained in [BH] and [T], we can state that $H(q,1,1,1)$ is homeomorphic to the 2−fold covering of the sphere branched over a well-specified three-bridge link (in the general case) directly obtainable from the diagram $\mathcal{G}(q,1,1,1)$. The symmetry axes fixed by one of the involutions of $\mathcal{G}(q,1,1,1)$ become the bridges of the branch set. Note that even if for $q=1$ we have a slightly different Heegaard diagram with respect to $\mathcal{G}(q,1,1,1)$, $q>1$, in Figure 4, the branch set of Figure 5 holds for all the non-negative values of q. In Figures 6-11 are depicted Heegaard diagrams and branch sets for the others homology spheres. □

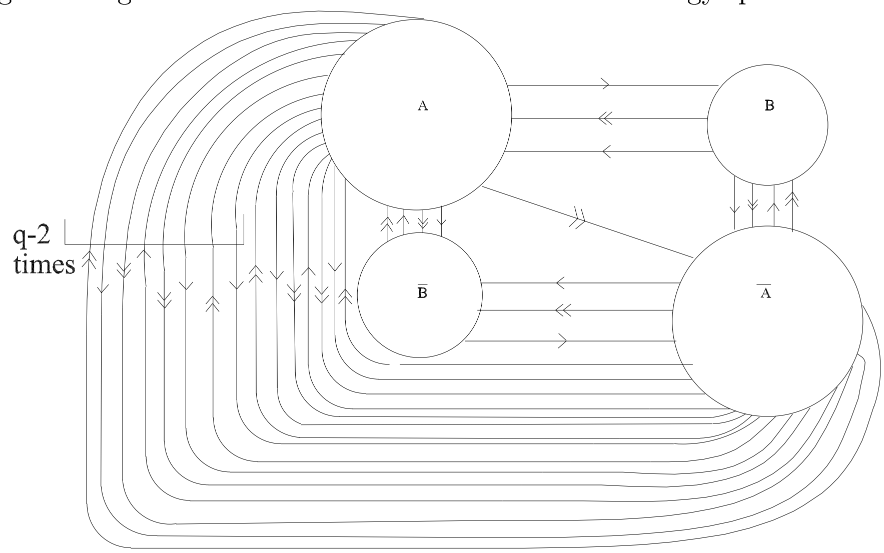

Figure 4. A Heegaard diagram of $H(q,1,1,1)$, for $q\geq 2$

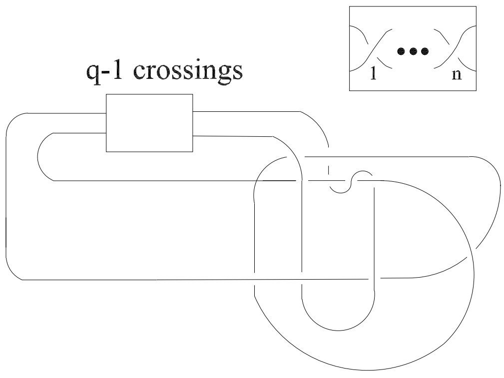

Figure 5. The branch set $K(q,1,1,1)$

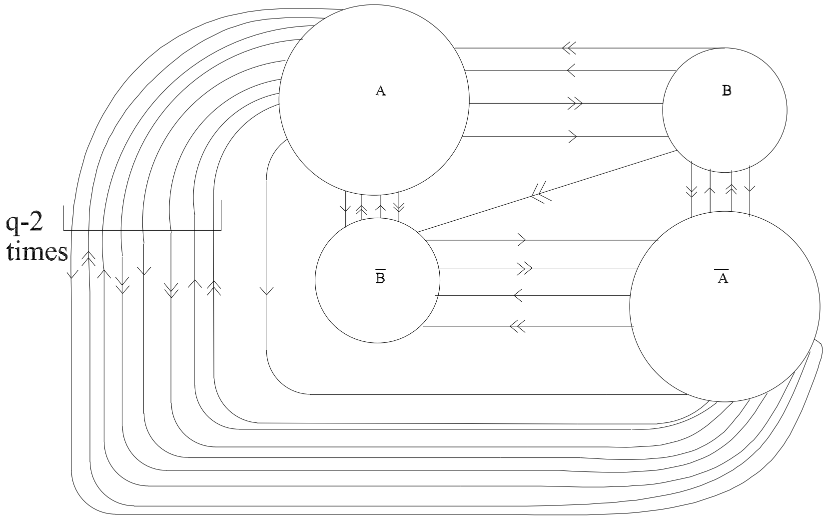

Figure 6. A Heegaard diagram of $H(q,1,-1,1)$, for $q \geq 2$

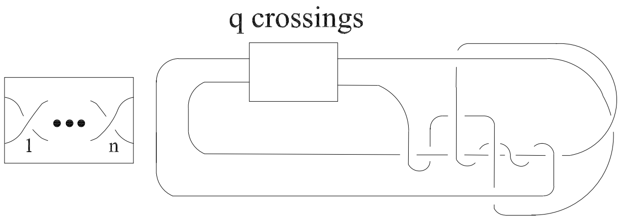

Figure 7. The branch set $K(q,1,-1,1)$

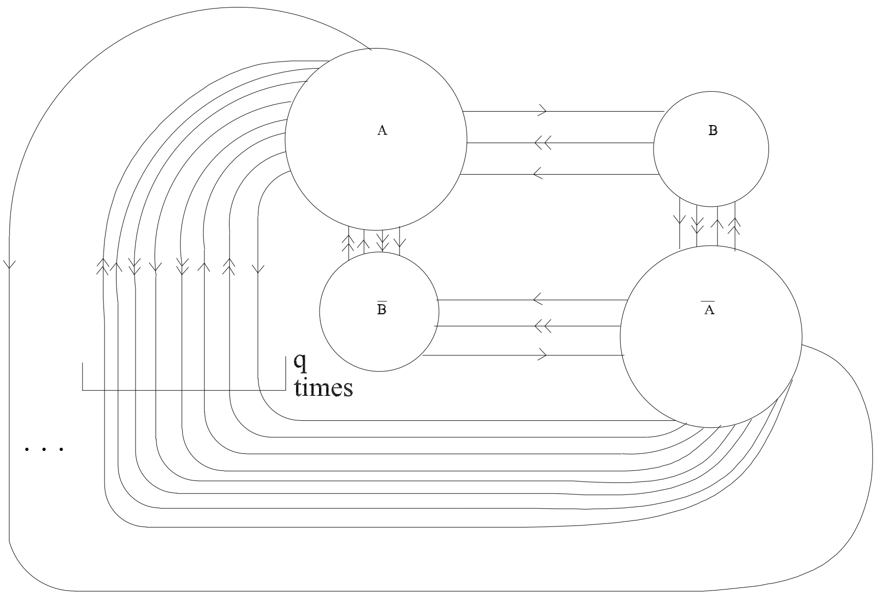

Figure 8. A Heegaard diagram of $H(q,-1,-1,1)$, for $q \geq 2$

Remark. The fundamental groups of the homology spheres $H(q, l, m, r)$ turn out to be pairwise isomorphic, since they admit equivalent presentations. In particular, $\pi(q, 1, 1, 1) \cong \pi(q, -1, -1, -1)$, $\pi(q, 1, -1, 1) \cong \pi(q, -1, 1, -1)$, $\pi(q, -1, -1, 1) \cong \pi(q, 1, 1, -1)$ and $\pi(q, -1, 1, 1, 1) \cong \pi(q, 1, -1, 1, -1)$.

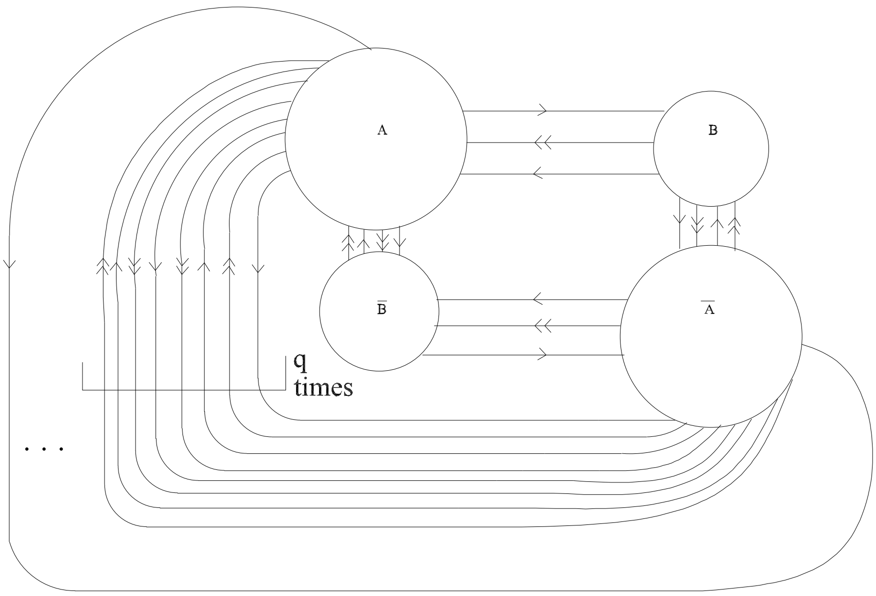

Figure 8. A Heegaard diagram of $H(q, -1, -1, 1)$, for $q \geq 2$

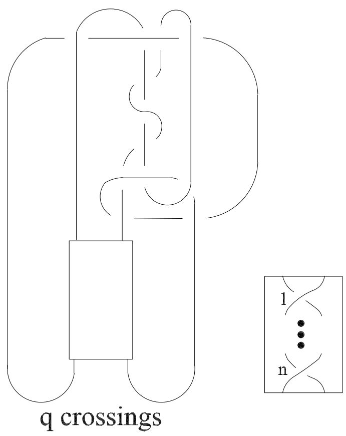

Figure 9. The branch set $K(q, -1, -1, 1)$

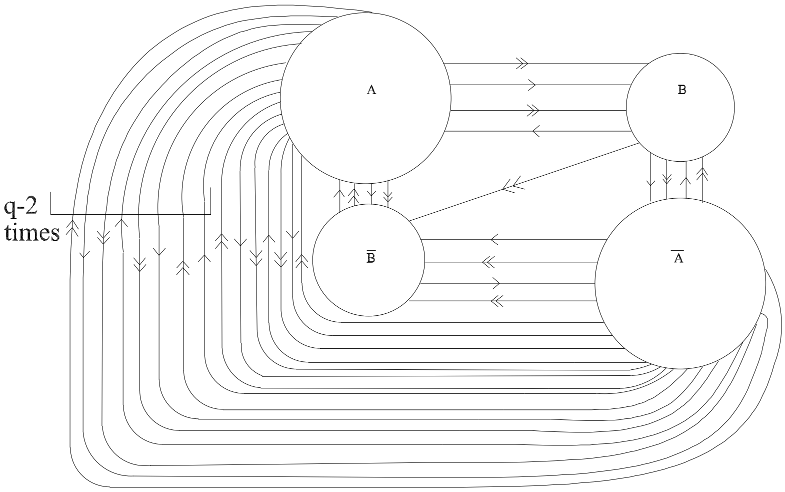

Figure 10. A Heegaard diagram of $H(q,-1,1,1)$, for $q \geq 2$

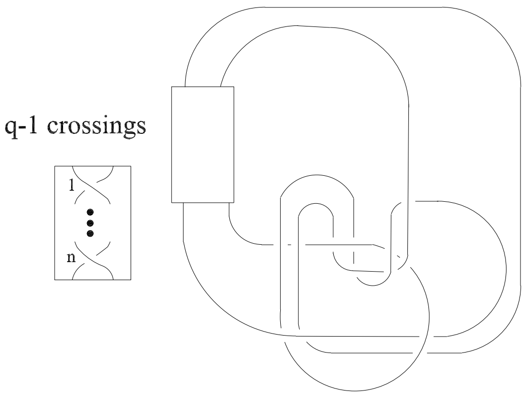

Figure 11. The branch set $K(q,-1,1,1)$

By applying Wirtinger algorithm to our branch sets, we obtain presentations for the knot groups $\pi(q,l,m,r) \cong \pi_1(\mathbb{S}^3 \setminus K(q,l,m,r))$ with two generators. Hence these knots are prime by Norwood ([N]). Recall that for a real number x, the symbol $\lfloor x \rfloor$ indicates the integer part of x, that is, the largest integer not greater than x.

Theorem 10. *The knots $K(q,l,m,n)$ are prime, and the knot group $\pi(q,l,m,r)$ admits the following presentation*

$$\pi(q,l,m,n) = \langle a,b\colon\ b^{\tau}[a^2,b^2]^{\lfloor \frac{q}{2} \rfloor} w [b^{-2}a^{-2}]^{\lfloor \frac{q}{2} \rfloor} \rangle$$

where

$$\tau = \begin{cases} -5 & l = m = 1 \\ -7 & l = 1,\ m = -1 \\ 3 & l = -1,\ m = 1,\ q > 1 \\ 1 & \textit{otherwise} \end{cases}$$

and

$$w = \begin{cases} a^3 & l = 1,\ q \textit{ odd} \\ b^2ab^2 & l = 1,\ q \textit{ even} \\ a^2b^2a^{-1}b^2a^2 & l = -1,\ q \textit{ odd} \\ a & l = -1,\ q \textit{ even} \end{cases}$$

Remark Note that the knot $K(1,1,1,1)$ is equivalent to the torus knot $T(3,5)$ and the knot $K(1,-1,1,1)$ is equivalent to the torus knot $T(3,7)$; moreover, the knots $K(1,-1,1,1)$ and $K(1,-1,-1,1)$ are equivalent to the $(-2,3,7)$-pretzel knot. This means that the manifold H(1,1,1,1) (resp. H(1,-1,1,1)) is homeomorphic to the Brieskorn manifold $M(3,5,2)$ (resp. $M(3,7,2)$) in the classic notation of [M].

References

[A] D. Auckly, *Two Fold Branched Covers*, preprint.

[B] J. Berge, *Embedding the exteriors of one-tunnel knots and links in the 3-sphere*, preprint.

[BH] J.S. Birman, H.M. Hilden, *Heegaard splittings of branched coverings of* $\mathbb{S}^3$, Trans. Amer. Math. Soc., **213** (1975), 315-352.

[CST] A. Cavicchioli, F. Spaggiari and A.I. Telloni, *Fundamental group and Covering Properties of Hyperbolic Surgery Manifolds*, Geometry, Volume 2013 (2013), ID 484508, http://dx.doi.org/10.1155/2013/484508.

[GL] C. McA. Gordon, J. Luecke, *Knots are determined by their complements*, J. Amer. Math. Soc. **2** (1989), 371-415.

[KK] S.H. Kim., Y. Kim, *On hyperbolic 3-manifolds obtained by Dehn surgery on links*, Intern. J. of Mathematics and Math. Sciences, Volume 2010, ID 573403, doi:10.1155/2010/573403.

[MOS] H. Matsuda, M. Ozawa, K. Shimokawa, *On non-simple reflexive links*, J. Knot Theory and iits Ram. **11** (2002), 787-791.

[M] J. Milnor, *On the 3dimensional Brieskorn manifolds M(p, q, r)*, Ann. Math. Studies **84** (1975), 175-225.

[N] F. H. Norwood, *Every two-generator knot is prime*, Proceed. of the Amer. Math. Soc. **86** (1982) (1), 143-147.

[O] M. Ochiai, *Heegaard diagrams of 3-manifolds*, Trans. Amer. Math. Soc., **328** (1991), 863-879.

[OS] R.P. Osborne, R. Stevens, Group presentations corresponding to spines of 3-manifolds. II, Trans. Amer. Math. Soc., **234** (1) (1977), 213-243

[R] D. Rolfsen, *Knots and Links*, Math. Lect. Ser., **7**, Publish or Perish, Berkeley, 1976.

[T] M. Takahashi, *Two knots with the same 2fold branched covering space*, Yokohama Math. J., **25** (1) (1977), 9199.

[Te] , M. Teragaito, *Lens with surgery yielding the 3-sphere*, J. Knot Theory Ramifications, **11**, 105 (2002).

Marina De Simone
Dipartimento di Matematica "Giuseppe Peano"
Via Carlo Alberto 10
Torino, Italy

Agnese Ilaria Telloni
Dipartimento di Ingegneria Industriale e Scienze Matematiche
Università Politecnica delle Marche
Via Brecce Bianche
Ancona, Italy