SPECIAL MATCHINGS CALCULATE THE PARABOLIC KAZHDAN–LUSZTIG POLYNOMIALS OF THE UNIVERSAL COXETER GROUPS

AGNESE ILARIA TELLONI

Abstract. In this paper we prove that the parabolic Kazhdan–Lusztig polynomials and the parabolic R-polynomials of the universal Coxeter group can be computed in a combinatorial way, by using special matchings.

1. Introduction

The Kazhdan–Lusztig polynomials $P_{u,v}(q)$ have been introduced in [8] and later studied in many context for their remarkable applications, mainly in representation theory and in the topology of Schubert varieties. They are polynomials in one variable q depending on two elements u and v of a Coxeter group W. In [8] Kazhdan and Lusztig introduced also the family of the Kazhdan–Lusztig R-polynomials $R_{u,v}(q)$, R-polynomials in brief, whose knowledge is equivalent to the knowledge of the family $\{P_{u,v}(q)\}_{u,v \in W}$. The following conjecture, known as the *Combinatorial Invariance Conjecture*, concerns equivalently the Kazhdan–Lusztig polynomials and the R-polynomials and was formulated independently by Lusztig, in private, and Dyer [5].

Conjecture 1.1. *The Kazhdan–Lusztig polynomial $P_{u,v}(q)$ and the R-polynomial $R_{u,v}(q)$ depend only on the combinatorial structure of the interval $[u,v]$ as a poset under the Bruhat order.*

The Combinatorial Invariance Conjecture means that if two intervals $[u,v]$ and $[u',v']$ (with respect to the Bruhat order) have the same isomophism type, hence $R_{u,v}(q) = R_{u',v'}(q)$ and $P_{u,v}(q) = P_{u',v'}(q)$. In [2] it was proved that Kazhdan–Lusztig and R-polynomial $R_{u,v}(q)$ and $P_{u,v}(q)$ can be computed from the knowledge of the interval $[e,v]$ (e denotes the identity element of W) via a combinatorial tool named *special matching*. A special matching of an element $v \in W$ is an involution of the lower Bruhat interval $[e,v]$ satisfying certain properties relating to the poset structure (see Section 2 for the exact definition). As a consequence, Conjecture 1.1 is true when $u = e$.

In [4], Deodhar defined two parabolic extensions of both the Kazhdan–Lusztig polynomials and the R-polynomials. Given a Coxeter system (W,S), a subset H of S, and $x \in \{q,-1\}$, the *parabolic Kazhdan–Lusztig and R-polynomial* are polynomials $P^{H,x}_{u,v}(q)$ and $R^{H,x}_{u,v}(q)$ indexed by elements u, v in the set W^H of minimal coset representatives. If

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$H = \emptyset$, the parabolic Kazhdan–Lusztig and R-polynomials coincide with the ordinary ones.

Recently Marietti generalized the main result in [2] to the parabolic setting, in the case (W, S) is a doubly laced Coxeter system, or a dihedral Coxeter system. The key tool of his proof is the concept of *H-special matching.* Given an element $w \in W^H$, a special matching M of $[e, w]$ is said *H-special* if it satisfies the property

$$u \leq w, \; u \in W^H, \; M(u) \lhd u \Rightarrow M(u) \in W^H.$$

The main result of [9] is that the H-special matchings of $w \in W^H$ can be used to calculate the parabolic Kazhdan–Lusztig and R-polynomials for the doubly laced Coxeter groups and the dihedral Coxeter groups.

In this work, we prove that the H-special matchings calculate the parabolic Kazhdan–Lusztig and R-polynomials for the universal Coxeter groups. Since it is known for the doubly laced Coxeter groups (when the order of the product of any two generators is ≤ 4), we are providing here the antipodal case (when the order of the product of any two generators is ∞). We prove the following recursive formula: if (W, S) is a Coxeter system for the universal Coxeter group, $H \subseteq S$, $u, w \in W^H$, and M is a H-special matching of w, then, for $u \leq w$

$$R^{H,x}_{u,w} = \begin{cases} R^{H,x}_{M(u),M(w)}(q), & \text{if } M(u) \rhd u, \\ (q-1)R^{H,x}_{u,M(w)}(q) + qR^{H,x}_{M(u),M(w)}(q), & \text{if } M(u) \lhd u \text{ and } M(u) \in W^H, \\ (q-1-x)R^{H,x}_{u,M(w)}(q), & \text{if } M(u) \lhd u \text{ and } M(u) \notin W^H \end{cases}$$

and $R^{H,x}_{u,w} = 0$ for $u \not\leq w$.

2. Basic definitions and preliminaries

2.1. **Coxeter systems.** Following [1], we recall some notations about Coxeter groups. A *Coxeter system* is a couple (W, S), where W is a Coxeter group and S a set of involutory generators for a suitable presentation of W. Each Coxeter group W is a partial ordered set by the Bruhat order, which will be indicated by $\leq$ troughout the paper. The rank function of W is the *length* of the elements, that is the number of generators in a reduced expression. We will denote the length of w as $L(w)$. A useful characterization of the Bruhat order is the following *Subword Property.*

Theorem 2.1. *([1, §2.1]) Let $w = s_1 s_2 \cdots s_q$ be a reduced expression. Then, $u \leq w$ if and only if there exists a reduced expression $u = s_{i_1} s_{i_2} \cdots s_{i_k}$, for $1 \leq i_1 < \cdots < i_k \leq q$.*

Given an element w in W, we call *left descent of* w a generator $s \in S$ such that $L(sw) < L(w)$, or equivalently, such that there is an expression of w beginning by s; analogously, s is a *right descent of* w if $L(ws) < L(w)$, or equivalently, if there is an expression of w ending by s. We denote respectively by $D_L(w)$ and $D_R(w)$ the sets of left and right descents of w.

For a Coxeter system (W, S) and a subset $J \subseteq S$, let W_J denote the *parabolic subgroup* of W generated by J, and let W^J and ${}^J W$ denote the sets of right and left minimal coset representatives, $W^J = \{w \in W \; : \; D_R(w) \subseteq S \setminus J\}$ and ${}^J W = \{w \in W \; : \; D_L(w) \subseteq$

$S \setminus J\}$. By [1, §2.4], each element $w \in W$ admits a unique decomposition, which has two mirrored versions:

$$w = w^J \cdot w_J,$$

where $w^J \in W^J$, $w_J \in W_J$ and $L(w) = L(w^J) + L(w_J)$ and, symmetrically

$$w = {}_J w \cdot {}^J w,$$

where ${}_J w \in W_J$, ${}^J w \in {}^J W$ and $L(w) = L({}_J w) + L({}^J w)$.

Given $u, w \in W$, we say that w *covers* u, or equivalently u *is covered by* w, denoted by $u \lhd w$ or $w \rhd u$, if $u \leq w$ and $L(w) = L(u) + 1$. In particular, u can be obtained by removing a single reflection in the reduced expression of w.

2.2. **Universal Coxeter groups.** For each positive integer n, the *universal Coxeter group* of *rank* n is presented by n generators of order 2 and no other relations, that is

$$W = \langle s_1, \ \ldots, \ s_n \colon {s_1}^2 = \ \cdots = \ {s_n}^2 = 1 \rangle.$$

Note that each element in a universal Coxeter group has a unique reduced expression, a unique left descent and a unique right descent. This implies that for an element of this group the word length of a word without consecutive repetitions of the same generator coincides with the Coxeter length.

2.3. **Special matchings.** A *special matching* of $w \in W$ is an involution M of the lower Bruhat interval $[e, w]$ such that either $u \lhd M(u)$ or $M(u) \lhd u$ for all $u \leq w$ and

$$u \lhd v \Longrightarrow M(u) \leq M(v) \text{ or } M(u) = v$$

for all elements $u, v \leq w$. Our notations and conventions concerning special matchings follow those of [9]. If $s \in D_L(w)$ (resp. $D_R(w)$), the involution λ_s (resp. ρ_s) defined by $\lambda_s(u) = su$ (resp. $\rho_s(u) = us$) for all $u \leq w$ is a special matching of $[e, w]$ (see [2, §2]) and we call it a *left multiplication matching* (resp. *right multiplication matching*).
Given two matchings M and N of $w \in W$ and $u \leq w$, we denote by $\langle M, N \rangle(u)$ the orbit of u under the action of the subgroup of the symmetric group on the interval $[e, w]$ generated by M and N.
An interval $[u, v]$ in a Coxeter group W is said to be *dihedral* if it is isomorphic (as a poset) to a finite *dihedral Coxeter group*, that is a Coxeter group with two generators.

Lemma 2.2. *(*[2, Lemmas 2.1, 4.1]*) Let (W, S) be a Coxeter system.*

1. *Let M be a special matching of W and $u, v \in W$ such that $M(v) \lhd v$ and $M(u) \rhd u$. Then M restricts to a special matching of the interval $[u, v]$.*
2. *Let M and N be two special matchings of W. Then, for all $u \in W$, the orbit $\langle M, N \rangle(u)$ is a dihedral interval (see Figure 1).*

Let w be an element in a Coxeter group W. It is well known that the intersection of the lower Bruhat interval $[e, w]$ with the dihedral parabolic subgroup $W_{\{s,t\}}$ generated by any two elements $s, t \in S$ has a maximal element; for short, we will denote it with $w_0(s, t)$.

The following definition is due to Marietti. It first appeared in an unpublished paper of 2013, and then in [9] and (in a slightly modified equivalent version) in [3]. A *right system for w* is a quadruple (J, s, t, M_{st}) such that:

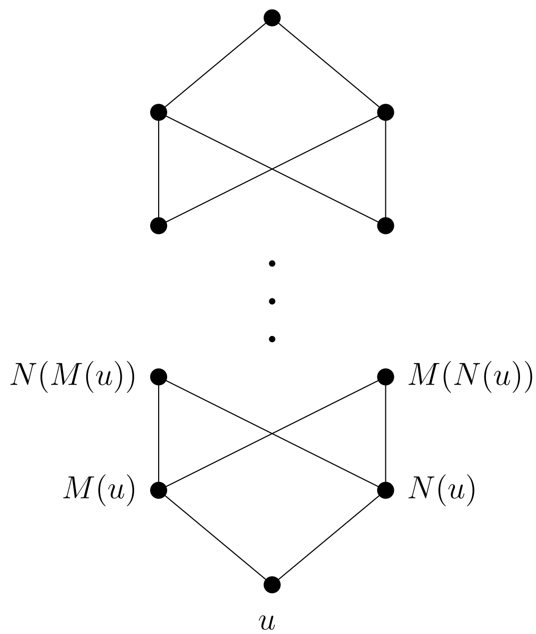

FIGURE 1. The orbit $\langle M, N\rangle(u)$.

R1. $J \subseteq S$, $s \in J$, $t \in S \setminus J$, and M_{st} is a special matching of $w_0(s,t)$ such that $M_{st}(e) = s$ and $M_{st}(t) = ts$;

R2. $(u^J)^{\{s,t\}} \cdot M_{st}\Big((u^J)_{\{s,t\}} \cdot {}_{\{s\}}(u_J)\Big) \cdot {}^{\{s\}}(u_J) \leq w$, for all $u \leq w$;

R3. if $r \in J$ and $r \leq w^J$, then r and s commute;

R4. (a) if $s \leq (w^J)^{\{s,t\}}$ and $t \leq (w^J)^{\{s,t\}}$, then $M_{st} = \rho_s$,
(b) if $s \leq (w^J)^{\{s,t\}}$ and $t \not\leq (w^J)^{\{s,t\}}$, then M_{st} commutes with λ_s,
(c) if $s \not\leq (w^J)^{\{s,t\}}$ and $t \leq (w^J)^{\{s,t\}}$, then M_{st} commutes with λ_t;

R5. if $v \leq w$ and $s \leq {}^{\{s\}}(v_J)$, then M_{st} commutes with ρ_s on $[e,v] \cap [e, w_0(s,t)] = [e, v_0(s,t)]$.

Symmetrically, a *left system for* w is a quadruple (J, s, t, M_{st}) such that:

L1. $J \subseteq S$, $s \in J$, $t \in S \setminus J$, and M_{st} is a special matching of $w_0(s,t)$ such that $M_{st}(e) = s$ and $M_{st}(t) = st$;

L2. $({}_J u)^{\{s\}} \cdot M_{st}\Big(({}_J u)_{\{s\}} \cdot {}_{\{s,t\}}({}^J u)\Big) \cdot {}^{\{s,t\}}({}^J u) \leq w$, for all $u \leq w$;

L3. if $r \in J$ and $r \leq {}^J w$, then r and s commute;

L4. (a) if $s \leq {}^{\{s,t\}}({}^J w)$ and $t \leq {}^{\{s,t\}}({}^J w)$, then $M_{st} = \lambda_s$,
(b) if $s \leq {}^{\{s,t\}}({}^J w)$ and $t \not\leq {}^{\{s,t\}}({}^J w)$, then M_{st} commutes with ρ_s,
(c) if $s \not\leq {}^{\{s,t\}}({}^J w)$ and $t \leq {}^{\{s,t\}}({}^J w)$, then M_{st} commutes with ρ_t;

L5. if $v \leq w$ and $s \leq ({}_J v)^{\{s\}}$, then M_{st} commutes with λ_s on $[e,v] \cap [e, w_0(s,t)] = [e, v_0(s,t)]$.

Given a right (resp. left) system (J, s, t, M_{st}) for w, the *matching associated with it* is the matching M acting as follows:

$$M(u) = (u^J)^{\{s,t\}} \cdot M_{st}\Big((u^J)_{\{s,t\}} \cdot {}_{\{s\}}(u_J)\Big) \cdot {}^{\{s\}}(u_J),$$

(resp. $M(u) = ({}_J u)^{\{s\}} \cdot M_{st}\Big(({}_J u)_{\{s\}} \cdot {}_{\{s,t\}}({}^J u)\Big) \cdot {}^{\{s,t\}}({}^J u))$, for all $u \leq w$. It is proved that this is a matching of w.

Theorem 2.3. *([3]) Let (W,S) be a Coxeter system and $w \in W$. Then*

1. *the matching associated with a (right or left) system of w is special;*
2. *a special matching of w is the matching associated with a (right or left) system of w.*

2.4. **Parabolic Kazhdan–Lusztig polynomials.** Let (W,S) be a Coxeter system, $H \subseteq S$, $w \in W^H$, $s \in D_L(w)$, $\lambda_s(w) \in W^H$; for all $u \leq w$, the parabolic Kazhdan–Lusztig polynomial $R^{H,x}_{u,w}(q)$ satisfies the following recursive formula:

(2.1)
$$R^{H,x}_{u,w}(q) = \begin{cases} R^{H,x}_{\lambda_s(u),\lambda_s(w)}(q), & \text{if } s \in D_L(u), \\ (q-1)R^{H,x}_{u,\lambda_s(w)}(q) + qR^{H,x}_{\lambda_s(u),\lambda_s(w)}(q), & \text{if } s \notin D_L(u) \text{ and } \lambda_s(u) \in W^H, \\ (q-1-x)R^{H,x}_{u,\lambda_s(w)}(q), & \text{if } s \notin D_L(u) \text{ and } \lambda_s(u) \notin W^H \end{cases}$$

and $R^{H,x}_{u,w}(q) = 0$ for $u \not\leq w$. Recall the following definition due to Marietti. A special matching M of an element $w \in W^H$ is *H-special* if, for all $u \leq w$, $u \in W^H$ it holds

$$M(u) \lhd u \Longrightarrow M(u) \in W^H.$$

By definition, the left multiplication matchings are H-special. An H-special matching M of w *calculates* the parabolic Kazhdan–Lusztig R-polynomials, or it is *calculating*, for short, if, for all $u \leq w$, the following holds:

(2.2)
$$R^{H,x}_{u,w}(q) = \begin{cases} R^{H,x}_{M(u),M(w)}(q), & \text{if } M(u) \lhd u, \\ (q-1)R^{H,x}_{u,M(w)}(q) + qR^{H,x}_{M(u),M(w)}(q), & \text{if } M(u) \rhd u \text{ and } M(u) \in W^H, \\ (q-1-x)R^{H,x}_{u,M(w)}(q), & \text{if } M(u) \rhd u \text{ and } M(u) \notin W^H. \end{cases}$$

In particular, all left multiplication matchings are calculating.

Theorem 2.4. *([10]) Given a Coxeter system (W,S) and $H \subseteq S$, let $w \in W^H$ and M be an H-special matching of w. Suppose that*

- *every H-special matching of v is calculating, for all $v \in W^H$, $v < w$,*
- *there exists a calculating special matching N of w such that $|\langle M,N\rangle(u)|$ divides $|\langle M,N\rangle(w)|$, for all $u \leq w$.*

Then M is calculating.

Recall that, for a *doubly laced Coxeter system* (W,S) the relations are of the form $s^2 = 1$ and $(ss')^{m(s,s')} = 1$, where $m(s,s') \leq 4$ for every $s, s' \in S$.

Two of the main results in [9] are the following:

Theorem 2.5. *([9, Theorem 4.5]) Let (W,S) be a doubly laced Coxeter system, $H \subseteq S$, w be any arbitrary element of W^H. Then every H-special matching of w calculates the $R^{H,x}$-polynomials.*

Theorem 2.6. *([9, Theorem 4.8]) Let (W,S) be a Coxeter system, $H \subseteq S$, $w \in W^H$ such that $[e,w]$ is a dihedral interval. Then every H-special matching of w calculates the $R^{H,x}$-polynomials.*

Note that the previous theorem implies that the H-special matchings calculate the parabolic R-polynomials of dihedral Coxeter groups.

3. Some properties of parabolic R-polynomials for universal Coxeter groups

In this section, we let (W,S) be a universal Coxeter system and $H \subseteq S$. We give some results about parabolic R-polynomials that are needed in the proof of the next section.

Notation 3.1. *In the sequel, we will often be considering the two generators s and t, and the elements of $W_{\{s,t\}}$ of a fixed Coxeter length. For the sake of simplicity, we denote by ℓ_k and $\bar{\ell}_k$ the elements $\ell\bar{\ell}\ell\dots$ and $\bar{\ell}\ell\bar{\ell}\dots$ of length k, where $\ell,\bar{\ell} \in \{s,t\}$, $\ell \neq \bar{\ell}$. In particular, we denote by s_k and t_k the elements sts... and tst... of length k.*

Lemma 3.2. *Let $u,w \in W^H$, $u \leq w$, $u = \ell_k y'$, $w = \ell_n y$, where $k < n$, $s,t \notin D_L(y)$, $D_L(y')$. Then:*

$$R^{H,x}_{u,w} = \sum_{i=0}^{h} q^i (q-1) R^{H,x}_{y',p_{n-k-(2i+1)}y} + q^{h+1} R^{H,x}_{r_{h+1}y',\bar{r}_{n-k-(h+1)}y}, \tag{3.1}$$

for all nonnegative integers $h \leq \frac{n-k-1}{2}$, where

$$\begin{cases} p = r = \ell & \text{if } k \text{ is odd and } h \text{ is odd} \\ p = \ell \text{ and } r = \bar{\ell} & \text{if } k \text{ is odd and } h \text{ is even} \\ p = r = \bar{\ell} & \text{if } k \text{ is even and } h \text{ is odd} \\ p = \bar{\ell} \text{ and } r = \ell & \text{if } k \text{ is even and } h \text{ is even.} \end{cases}$$

Proof. We prove the statement by induction on h; let us consider the case k even, the other ones are analogous. For $h = 0$, we get

$$R^{H,x}_{u,w} = R^{H,x}_{y',\ell_{n-k}y} = (q-1) R^{H,x}_{y',\bar{\ell}_{n-k-1}y} + q R^{H,x}_{\ell y',\bar{\ell}_{n-k-1}y},$$

where the first equality holds by the formula (2.1) and since $L(u) = k + L(y')$, $L(w) = k + L(\ell_{n-k}y)$. Indeed, the second equality holds since $\ell \notin D_L(y)$. Let now $h > 0$; by applying the inductive hypothesis, we get:

$$R^{H,x}_{u,w} = R^{H,x}_{y',\ell_{n-k}y} = \sum_{i=0}^{h} q^i(q-1)R^{H,x}_{y',\bar{\ell}_{n-k-(2i+1)}y} + q^{h+1}R^{H,x}_{r_{h+1}y',\bar{r}_{n-k-(h+1)}y}$$

$$= \sum_{i=0}^{h} q^i(q-1)R^{H,x}_{y',\bar{\ell}_{n-k-(2i+1)}y} + q^{h+1}(q-1)R^{H,x}_{r_{h+1}y',r_{n-k-(h+2)}y} + q^{h+2}R^{H,x}_{\bar{r}_{h+2}y',r_{n-k-(h+2)}y}$$

$$= \sum_{i=0}^{h} q^i(q-1)R^{H,x}_{y',\bar{\ell}_{n-k-(2i+1)}y} + q^{h+1}(q-1)R^{H,x}_{y',\bar{\ell}_{n-k-(2h+3)}y} + q^{h+2}R^{H,x}_{\bar{r}_{h+2}y',r_{n-k-(h+2)}y},$$

that is, for h even:

$$\sum_{i=0}^{h+1} q^i(q-1)R^{H,x}_{y',\bar{\ell}_{n-k-(2i+1)}y} + q^{h+2}R^{H,x}_{\bar{\ell}_{h+2}y',\ell_{n-k-(h+2)}y},$$

and for h odd:

$$\sum_{i=0}^{h+1} q^i(q-1)R^{H,x}_{y',\bar{\ell}_{n-k-(2i+1)}y} + q^{h+2}R^{H,x}_{\ell_{h+2}y',\bar{\ell}_{n-k-(h+2)}y}.$$

□

Analogously, we obtain the following Lemma:

Lemma 3.3. *Let $u', w \in W^H$, $u' \leq w$, $u' = \bar{\ell}_k y'$, $w = \ell_n y$, where $k < n$, $s, t \notin D_L(y), D_L(y')$. Then:*

$$R^{H,x}_{u',w} = \sum_{i=0}^{h} q^i(q-1)R^{H,x}_{y',p_{n-k-(2i+1)}y} + q^{h+1}R^{H,x}_{r_{k+h+1}y',\bar{r}_{n-(h+1)}y}, \tag{3.2}$$

for all nonnegative integers $h \leq \frac{n-k-1}{2}$, where

$$\begin{cases} p = r = \ell & \text{if } k \text{ is odd and } h \text{ is even} \\ p = \ell \text{ and } r = \bar{\ell} & \text{if } k \text{ is odd and } h \text{ is odd} \\ p = r = \bar{\ell} & \text{if } k \text{ is even and } h \text{ is odd} \\ p = \bar{\ell} \text{ and } r = \ell & \text{if } k \text{ is even and } h \text{ is even.} \end{cases}$$

Let us call the polynomial $q^{h+1}R^{H,x}_{r_{h+1}y',\bar{r}_{n-k-(h+1)}y}$ in (3.1) (resp. $q^{h+1}R^{H,x}_{r_{k+h+1}y',\bar{r}_{n-(h+1)}y}$ in (3.2)), for $h = \max\{0, \lfloor\frac{n-k-1}{2}\rfloor\}$, the *rest* of $R^{H,x}_{u,w}$ (resp. $R^{H,x}_{u',w}$) and denote it with $r(x)$ (resp. $r'(x)$).

Remark 3.4. *The previous lemmas hold in a more general context with respect than the universal Coxeter groups, even if it could require a little modification of the proofs. For example, Lemma 3.2 holds for any Coxeter system (W, S), under the additional hypotheses that $u = \ell_k y'$ and $w = \ell_n y$ are reduced expressions, $k > 0$, $\bar{\ell}_b^{-1}u \in W^H$ for all nonnegative integers $b \leq k$, and s, t not both $\leq y$. In particular, if $\bar{r} \in D_L(r_{h+1}y)$, then $r(x) = 0$.*

Corollary 3.5. *Let $u, u', w \in W^H$, $u = \ell_k y'$, $u' = \bar{\ell}_k y'$, $w = \ell_n y$, $u, u' \leq w$, where $k < n$, $s, t \notin D_L(y)$, $D_L(y')$, and s, t not both $\leq y$. Hence the rests $r(x)$ and $r'(x)$ are such that:*

$$r(x) = \begin{cases} qR^{H,x}_{cy',y} & n-k=1 \\ 0 & n-k>1 \text{ odd} \\ q^{h+1}(q-1)R^{H,x}_{cy',y} & \text{otherwise,} \end{cases}$$

and

$$r'(x) = \begin{cases} qR^{H,x}_{cy',y} & (n,k)=(1,0) \\ 0 & (n,k)\neq(1,0) \text{ and } n-k \text{ odd} \\ q^{h+1}(q-1)R^{H,x}_{cy',y} & \text{otherwise,} \end{cases}$$

where $c = \bar{\ell}$ if k is odd and $c = \ell$ otherwise.

Proof. We sketch the proof in one of the mentioned cases, the other ones are analogous. Assume n even, k odd, $n-k > 1$ and h even (for example, $n = 10$, $k = 5$, hence $h = 2$). Then, by Equation (3.1), we get:

$$r(x) = q^{h+1}R^{H,x}_{\bar{\ell}_{h+1}y',\ell_{n-k-(h+1)}y} = q^{h+1}R^{H,x}_{\bar{\ell}_{h+1}y',\ell_h y} = 0, \tag{3.3}$$

where the second equality holds since the hypoteses imply $h = \frac{n-k-1}{2} \geq 2$ and the third one holds since s, t not both $\leq y$ implies $\bar{\ell}_{h+1}y' \not\leq \ell_h y$.
Analogously, by Equation (3.2), we get:

$$r'(x) = q^{h+1}R^{H,x}_{\ell_{k+h+1}y',\bar{\ell}_{n-(h+1)}y} = q^{h+1}R^{H,x}_{\ell_{\frac{n+k+1}{2}}y',\bar{\ell}_{\frac{n+k-1}{2}}y} = 0, \tag{3.4}$$

where the second equality holds since the hypoteses imply $h = \frac{n-k-1}{2} \geq 2$ and the third one holds since s, t not both $\leq y$ implies $\ell_{\frac{n+k+1}{2}}y' \not\leq \bar{\ell}_{\frac{n+k-1}{2}}y$.

□

Corollary 3.6. *Assume the hypotheses of Corollary 3.5. Hence*

$$R^{H,x}_{u,w} = R^{H,x}_{u',w},$$

except in the case $n-k=1$, $(n,k) \neq (1,0)$ and $cy' \leq y$, where $c = \begin{cases} \ell & k \text{ even} \\ \bar{\ell} & k \text{ odd.} \end{cases}$

Lemma 3.7. *Let u, w be as in Corollary 3.5 and let $w' = \bar{\ell}_n y \in W^H$. Hence*

$$R^{H,x}_{u,w} = R^{H,x}_{u,w'},$$

except possibly in the case $n-k=1$, $cy' \leq y$, for $c = \begin{cases} \ell & k \text{ even} \\ \bar{\ell} & k \text{ odd.} \end{cases}$

Proof. Assume $c = \begin{cases} \ell & k \text{ even} \\ \bar{\ell} & k \text{ odd} \end{cases}$, $\ell, \bar{\ell} \in \{s,t\}$, $\bar{\ell} \neq \ell$. If $n = k+1$, we get:

$$R^{H,x}_{u,w} = R^{H,x}_{\ell_k y', \ell_{k+1} y} = R^{H,x}_{y', cy} = (q-1) R^{H,x}_{y,y'} + q R^{H,x}_{cy',y},$$

which is equal to $R^{H,x}_{\ell_k y', \bar{\ell}_{k+1} y}$ if $cy' \not\leq y$.
Now we prove the result for $n - k \geq 2$ by induction on n. If $n = k+2$, we get:

$$R^{H,x}_{u,w} = R^{H,x}_{\ell_k y', \ell_{k+2} y} = R^{H,x}_{y', c\bar{c}y} = (q-1) R^{H,x}_{y', \bar{c}y} + q R^{H,x}_{cy', \bar{c}y} = \\ (q-1)^2 R^{H,x}_{y',y} + q(q-1)(R^{H,x}_{\bar{c}y',y} + R^{H,x}_{cy',y}) + q^2 R^{H,x}_{c\bar{c}y',y}.$$

On the other hand:

$$R^{H,x}_{u,w'} = R^{H,x}_{\ell_k y', \bar{\ell}_{k+2} y} = (q-1) R^{H,x}_{y', cy} + q(q-1) R^{H,x}_{\bar{c}y',y} \\ = (q-1)^2 R^{H,x}_{y',y} + q(q-1)(R^{H,x}_{\bar{c}y',y} + R^{H,x}_{cy',y}) + q^2 R^{H,x}_{c\bar{c}y',y},$$

where the term $R^{H,x}_{c\bar{c}y',y}$ is zero since c and $\bar{c}$ are not both $\leq y$. Let now $u = \ell_k y'$, $w = \ell_{n+1} y$, $w' = \bar{\ell}_{n+1} y$.

$$R^{H,x}_{u,w} = R^{H,x}_{\bar{\ell}_k y', \ell_{n+1} y} = (q-1) R^{H,x}_{\bar{\ell}_k y', \bar{\ell}_n y} + q R^{H,x}_{\ell_{k+1} y', \bar{\ell}_n y} = (q-1) R^{H,x}_{\ell_k y', \bar{\ell}_n y} + q R^{H,x}_{\bar{\ell}_{k+1} y', \bar{\ell}_n y} \\ = (q-1) R^{H,x}_{\ell_k y', \ell_n y} + q R^{H,x}_{\bar{\ell}_{k+1} y', \ell_n y} = R^{H,x}_{u,w'},$$

where the first and the third equalities hold by Corollary 3.6 and the fourth one holds by the inductive hypothesis. □

4. H-special matchings and parabolic R-polynomials for universal Coxeter groups

In this section, we prove the main result of the paper.

Let (W, S) be a universal Coxeter system, $H \subseteq S$ and $w \in W^H$. Let M be an H-special matching of w associated to a system (J, s, t, M_{st}).

Remark 4.1. *The restriction M_{st} of M to the dihedral interval $[e, w_0(s,t)]$ is locally a left or right multiplication matching. In fact, any element $u \in [e, w_0(s,t)]$ is either ℓ_k or $\bar{\ell}_k$, for a suitable k and $M(u)$ can be one of the following elements $\lambda_s(u)$, $\lambda_t(u)$, $\rho_s(u)$, $\rho_t(u)$.*

Let $p \in \{s,t\}$. Then M commutes with λ_p (resp. ρ_p) on $[u,v] \subset [e, w_0(s,t)]$ if and only if M does not coincide with $\lambda_{\bar{p}}$ (resp. $\rho_{\bar{p}}$) on any element $z \in [u,v]$.

Let $u = \ell_k$, $u < w_0(s,t)$. We say that u is *locally maximal for* (M, λ_ℓ) if $M(u) = \lambda_\ell(u) = \bar{\ell}_{k-1}$ and $M(\bar{\ell}_k) = \bar{\ell}_{k+1}$.

We state the following lemma.

Lemma 4.2. *Let $p \in \{s,t\}$ such that M_{st} commutes with the right multiplication matching ρ_p on $[e, w_0(s,t)]$. Let $\ell_k < w_0(s,t)$ be locally maximal for (M, λ_ℓ). Then M coincides with λ_ℓ on all elements in $\{\ell_{i+1}, \bar{\ell}_i \ : i = k-2h, \ldots, k-1\}$ and $M(\ell_{k-2h}) = \ell_{k-2h-1}$, for a suitable $h \in \mathbb{N}$.*

Proof. By the hypotheses and Remark 4.1, $M(\bar{\ell}_k) = \bar{\ell}_{k+1}$, M acts as ρ_p on $\bar{\ell}_k$ and $p \in D_R(\ell_k)$. This implies that, if j is the maximal number lower than k such that $M(\bar{\ell}_j) = \bar{\ell}_{j+1}$ (resp. $M(\ell_j) = \ell_{j+1}$), necessarily j has the same parity (resp. different parity) with respect to k; otherwise M_{st} would act as $\rho_{\bar{p}}$ on $\bar{\ell}_j$ (resp. ℓ_j) and it would not commute with ρ_p (see Remark 4.1). □

A *calculating chain for* w is a finite sequence $M = M^0 \to M^1 \to \cdots \to M^r$ of H-special matchings of w such that:

- M^i commutes with M^{i-1} for every $i \in \{1, \dots, r\}$;
- $M^i(w) \neq M^{i-1}(w)$ for every $i \in \{1, \dots, r\}$;
- M^r is calculating.

A *weak calculating chain for* w is a finite sequence $M = M^0 \to M^1 \to \cdots \to M^r$ of H-special matchings of w such that:

- $|\langle M^{i-1}, M^i\rangle(u)|$ divides $|\langle M^{i-1}, M^i\rangle(w)|$ for every $u \in [e, w]$, $i \in \{1, \dots, r\}$;
- M^r is calculating.

Note that each calculating chain for w is also a weak calculating chain for w, since when two matchings M and N for w commute, the orbit of every element $u \leq w$ under M and N has either 2 elements (if $M(u) = N(u)$) or 4 elements (otherwise).

We may now rephrase Theorem 2.4 as follows:

Proposition 4.3. *Given a Coxeter system (W, S) and $H \subseteq S$, let $w \in W^H$ and M be an H-special matching of w. Suppose that*

- *every H-special matching of v is calculating, for all $v \in W^H$, $v < w$,*
- *there exists a weak calculating chain $M = M^0 \to M^1 \to \cdots \to M^r$ for w.*

Then M is calculating.

Before stating the main theorem, we give the following simple lemma, which will be useful in the proof.

Lemma 4.4. *Let (W, S) be a universal Coxeter system, $H \subseteq S$, $\Sigma = (J, s, t, M_{st})$ be a left or right system for $w \in W^H$, $w = \ell_r y$, $s, t \notin D_L(y)$, and $c \in \{s, t\}$ such that M_{st} does not commute with ρ_c on $[e, w_0(s, t)]$. Then*

1. *if $c \in H$ then $\bar{c} \in H$;*
2. *if Σ is a left system, then $c \not\leq y$.*

Proof. By Remark 4.1, there is an element $z \in [e, w_0(s, t)]$ such that $M_{st}(z) = \rho_{\bar{c}}(z)$ and $M_{st}(z) \lhd z$. Hence $\bar{c}$ is the unique right descent for z. Now, if $\bar{c} \notin H$ (hence $z \in W^H$), necessarily $c \notin H$, since, by definition of H-special matching, $z \in W^H$ implies $M(z) \in W^H$. If Σ is a left system for w, by L4, then $c \not\leq^{\{s,t\}} ({}^J w) = y$. □

Theorem 4.5. *Let (W, S) be a universal Coxeter system, $H \subseteq S$ and w be any arbitrary element of W^H. Then every H-special matching of w calculates the $R^{H,x}$-polynomials.*

Proof. We proceed by induction on $L(w)$, the case $L(w) \leq 2$ being trivial.
Let M be an H-special matching of w. We assume that M is not a left multiplication, since left multiplication matchings are calculating by definition. If there exists a left

multiplication matching λ commuting with M and such that $\lambda(w) \neq M(w)$, we are done by Theorem 2.4. We assume that such a multiplication matching does not exist.

By Theorem 2.3, M is associated with a (right or left) system (J, s, t, M_{st}) for w. Set M^0 equal to M_{st}, let $p \in \{s, t\}$ be the right descent of $w_0(s, t)$ and $\bar{p}$ be the element of $\{s, t\}$ different from p; finally let n be such that $w_0(s, t) = \ell_n$. If $n \leq 4$, we can proceed as in the proof of [9, Theorem 4.5], as noticed in [9, Remark 3.4], hence assume $n > 4$. There will be two instances to consider:

a) M_{st} acts either as λ_ℓ or as ρ_p on both ℓ_{n-1} and $\bar{\ell}_{n-1}$;

b) M_{st} acts as a left multiplication on exactly one element of $\{\ell_{n-1}, \bar{\ell}_{n-1}\}$ and as a right multiplication on the other one.

First, we suppose that M is associated to a right system. If $(w^J)^{\{s,t\}} \neq e$, we can proceed as in the proof of [9, Theorem 4.5]; otherwise $w = \ell_r y$, $s \notin D_L(y)$, $t \not\leq y$ (note that $n \in \{r, r+1\}$). Moreover, we assume $y \neq e$, since H-special matchings are calculating for dihedral Coxeter groups by [9]. Under these hypoteses, (J, s, t, M_{st}) is a right system for w if and only if R1, R2 and R5 hold. Also, any element $u \leq w$ is of the form $c_k y'$, where $c \in \{s, t\}$, $s \notin D_L(y)$, $t \not\leq y'$ and $y' \leq y$, therefore $c_k = (u^J)_{\{s,t\}} \cdot {}_{\{s\}}(u_J)$ and $y' = {}^{\{s\}}(u_J)$. We have the following cases:

(1) $w_0(s, t) = \ell_r$ and either $s, t \in H$ or $s, t \notin H$;
(2) $w_0(s, t) = \ell_r$, $s \in H$, $t \notin H$ and $p = s$;
(3) $w_0(s, t) = \ell_r$, $s \in H$, $t \notin H$ and $p = t$;
(4) $\ell_r < w_0(s, t)$.

Case (1). If **a)** holds, define M^1 by the following top-to-bottom algorithm:

- $M^1(\ell_r) \neq M^0(\ell_r)$, $M^1(\ell_r) \lhd \ell_r$ and $M^1(M^0(\ell_r)) = M^0(M^1(\ell_r))$;
- for the element $\bar{\ell}_k$ of maximal length k, $k > 3$, such that $M^0(\bar{\ell}_k) = \lambda_{\bar{\ell}}(\bar{\ell}_k)$, set $M^1(\bar{\ell}_k) = \bar{\ell}_{k-1}$ and $M^1(\ell_{k-1}) = M^0(\bar{\ell}_{k-1})$, in which case $M^1(M^0(\bar{\ell}_k)) = M^0(M^1(\bar{\ell}_k))$ (see Figure 2). Then, repeat this step on $[e, \bar{\ell}_{k-2}]$ and so on until there are not elements $\bar{\ell}_b$, $b > 3$, on which M^0 coincides with $\lambda_{\bar{\ell}}$;
- for elements u not yet considered, set $M^1(u) = M^0(u)$.

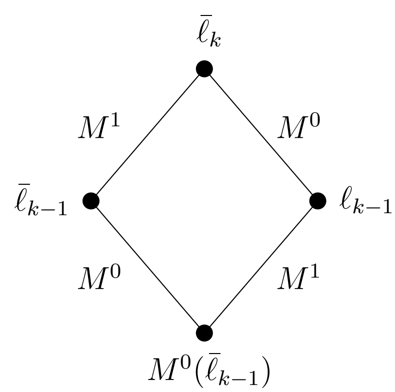

FIGURE 2. The H-special matchings M^0 and M^1 on $\bar{\ell}_k$.

Now, if M^1 does not commute with λ_ℓ, this happens since $M^1(tst) = st$ and $\ell = s$ (in fact, by construction, M^1 does not coincide with $\lambda_{\bar{\ell}}$ on any element $u \in [e, w_0(s,t)]$ of length greater than 3 - see Remark 4.1). In this case define M^2 so that:

- $M^2(\ell_r) \neq M^1(\ell_r)$, $M^2(\ell_r) \lhd \ell_r$ and $M^2(M^1(\ell_r)) = M^1(M^2(\ell_r))$;
- $M^2(st) = sts$ and $M^2(tst) = M^1(sts)$;
- for elements u not yet considered, set $M^2(u) = M^1(u)$.

If the last matching defined, say M^j, for $j \in \{1,2\}$, does not coincide with λ_ℓ on ℓ_r, then $C_{st} : M^0 \to \dots M^j \to \lambda_\ell$ is a calculating chain for $w_0(s,t)$; otherwise, define M^{j+1} so that:

- $M^{j+1}(\ell_r) \neq M^j(\ell_r)$, $M^{j+1}(\ell_r) \lhd \ell_r$ and $M^{j+1}(M^j(\ell_r)) = M^j(M^{j+1}(\ell_r))$;
- for elements u not yet considered, set $M^{j+1}(u) = M^j(u)$.

Now, $C_{st} : M^0 \to \cdots \to M^{j+1} \to \lambda_\ell$ is a calculating chain for $w_0(s,t)$. By Theorem 2.3 every H-special matching of $[e, w_0(s,t)]$ is associated to a (right or left) system; by construction, each matching M^i of C_{st} is associated to a right system (J, s, t, M^i) for $w_0(s,t)$, which is also a right system for w. In fact, by the simplification of the axioms above mentioned, we need to check only R1, R2 and R5. These properties hold since $w_0(s,t) = \ell_r$, M is a right H-special matching of w and by the construction of the other matchings of C_{st}. Hence, for each matching M^i of C_{st}, it is possible to consider the associated matching of w, which is H-special by Theorem 2.3. Thus C_{st} can be extended to a calculating chain of w, and we are done by Proposition 4.3.

If **b)** holds, define M^1 as follows. If $r = 5$:

- $M^1(\ell_5) \neq M^0(\ell_5)$, $M^1(\ell_5) \lhd \ell_5$, $M^1(M^0(\ell_5)) = \bar{\ell}_3$ and $M^1(\ell_3) = M^0(\bar{\ell}_3)$;
- for elements u not yet considered, set $M^1(u) = M^0(u)$.

Otherwise, if $r \geq 6$, define M^1 by the following top-to-bottom algorithm:

- $M^1(\ell_r) \neq M^0(\ell_r)$, $M^1(\ell_r) \lhd \ell_r$ and $M^1(M^0(\ell_r)) = \bar{\ell}_{r-2}$, $M^1(\ell_{r-2}) = x$, $x \neq M^0(\bar{\ell}_{r-2})$ and $x \lhd \ell_{r-2}$, $M^1(M^0(\bar{\ell}_{r-2})) = M^0(M^1(\ell_{r-2}))$;
- for the element $\bar{\ell}_k$ of maximal length k, $3 < k < r-3$, such that $M^0(\bar{\ell}_k) = \lambda_{\bar{\ell}}(\bar{\ell}_k)$, set $M^1(\bar{\ell}_k) = \bar{\ell}_{k-1}$ and $M^1(\ell_{k-1}) = M^0(\bar{\ell}_{k-1})$. Then, repeat this step on $[e, \bar{\ell}_{k-2}]$ and so on until there are not elements $\bar{\ell}_b$, $3 < b < r-3$, on which M^0 coincides with $\lambda_{\bar{\ell}}$;
- for elements u not yet considered, set $M^1(u) = M^0(u)$.

In both cases the cardinality of the orbit $\langle M^1, M^0\rangle(\ell_r)$ (which is 6 in the first case and 8 in the second one) is a multiple of $|\langle M^1, M^0\rangle(u)|$, for every $u < \ell_r$, $u \notin \langle M^1, M^0\rangle(\ell_r)$ (which is 2 in the first case, and 2 or 4 in the second case). Now, if M^1 commutes with λ_ℓ, we have a weak calculating chain $C_{st} : M^0 \to M^1 \to \lambda_\ell$ for $w_0(s,t)$. Otherwise, we note that, by construction, M^1 is an H-special matching for ℓ_r (associated to a right system), for which **a)** holds. Hence, we can proceed as in the previous case in order to obtain a weak calculating chain C_{st} for $w_0(s,t)$. As before, C_{st} can be extended to a weak calculating chain for w, and we are done by Proposition 4.3.

Case (2). If **a)** holds, we can proceed as in Case (1). If **b)** holds, necessarily $M(\ell_r) = \ell_{r-1}$ and $\bar{\ell}_{r-1}$ is locally maximal for $(M, \lambda_{\bar{\ell}})$. By Lemma 4.2, M coincides with $\lambda_{\bar{\ell}}$ on all elements in $\{\bar{\ell}_{i+1}, \ell_i \ : i = r-2h+1, \dots, r-2\}$ and $M^0(\bar{\ell}_{r-2h+1}) = \bar{\ell}_{r-2h}$, for a suitable $h \in \mathbb{N}$, $h \geq 2$. If $r = 2h+1$, which implies $\ell = s$, define M^1 so that:

- $M^1(s_j) = t_{j-1}$ for $j = 4, \ldots, r$ and $M^1(sts) = st$;
- for elements u not yet considered, set $M^1(u) = M^0(u)$.

Otherwise, define M^1 by the following top-to-bottom algorithm:

- $M^1(\ell_j) = \bar{\ell}_{j-1}$ for $j = r - 2h + 1, \ldots, r$;
- for the element $\bar{\ell}_k$ of maximal length k, $3 < k < r - 2h$, such that $M^0(\bar{\ell}_k) = \lambda_{\bar{\ell}}(\bar{\ell}_k)$, set $M^1(\bar{\ell}_k) = \bar{\ell}_{k-1}$ and $M^1(\ell_{k-1}) = M^0(\bar{\ell}_{k-1})$ (note that, by Lemma 4.2, $M^0(\bar{\ell}_{k-1}) = \ell_{k-2}$). Then, repeat this step on $[e, \bar{\ell}_{k-2}]$ and so on until there are not elements $\bar{\ell}_b$, $3 < b < r - 2h$, on which M^0 coincides with $\lambda_{\bar{\ell}}$;
- for elements u not yet considered, set $M^1(u) = M^0(u)$.

Now, in both cases the cardinality of the orbit $\langle M^1, M^0\rangle(\ell_r)$ (which is $4h+2$ in the first case and $4h+4$ in the second case) is a multiple of the cardinality $|\langle M^1, M^0\rangle(u)|$, for every $u < \ell_r$, $u \notin \langle M^1, M^0\rangle(\ell_r)$ (which is 2 in the first case and 2 or 4 in the second case). Moreover, M^1 commutes with λ_ℓ, since by construction it does not coincide with $\lambda_{\bar{\ell}}$ on any element $u \in [e, w_0(s,t)]$ (see Remark 4.1), but M^1 coincides with λ_ℓ on ℓ_r. Thus we define M^2 so that:

- $M^2(\ell_r) \neq M^1(\ell_r)$, $M^2(\ell_r) \lhd \ell_r$ and $M^2(M^1(\ell_r)) = M^1(M^2(\ell_r))$;
- for elements u not yet considered, set $M^2(u) = M^1(u)$.

Now, $C_{st} : M^0 \to M^1 \to M^2 \to \lambda_\ell$ is a weak calculating chain for $w_0(s,t)$. As before, by Theorems 2.3 and 2.3, it is possible to extend C_{st} to a calculating chain for w, and we are done by Proposition 4.3.

Case (3). By the hypotheses, necessarily $M(w) = \lambda_\ell(w)$ and M commutes with ρ_s on $[e, \ell_r]$ by Remark 4.1. Hence M calculates the R-polynomial $R^{H,x}_{u,w}(q)$ for all elements u such that $M(u) = \lambda_\ell(u)$, since λ_ℓ is calculating. Moreover, for $u \in [e, \ell_r]$, M is calculating by [9]. So we consider $u \in [e, w]$ such that $M(u) \neq \lambda_\ell(u)$, $u = c_k y'$, $k \leq r$, $c \in \{s,t\}$, $s \notin D_L(y')$, $t \not\leq y'$. The following situations can occur:

- $M(u) = \lambda_{\bar{\ell}}(u)$ and $M(u) \lhd u$. In this case $c = \bar{\ell}$ and $1 \leq k \leq r-2$. Hence
$$R^{H,x}_{u,w} = R^{H,x}_{\bar{\ell}_k y', \ell_r y} = R^{H,x}_{\ell_k y', \ell_r y} = R^{H,x}_{\bar{\ell}_{k-1} y', \bar{\ell}_{r-1} y} = R^{H,x}_{\ell_{k-1} y', \bar{\ell}_{r-1} y} = R^{H,x}_{M(u), M(w)},$$
where both the second and the fourth equalities hold by Corollary 3.6.
- $M(u) = \lambda_{\bar{\ell}}(u)$ and $M(u) \rhd u$. In this case $c = \ell$ and $0 \leq k \leq r-3$. Hence
$$\begin{aligned} R^{H,x}_{u,w} = R^{H,x}_{\ell_k y', \ell_r y} = R^{H,x}_{\bar{\ell}_k y', \ell_r y} &= (q-1) R^{H,x}_{\bar{\ell}_k y', \bar{\ell}_{r-1} y} + q R^{H,x}_{\ell_{k+1} y', \bar{\ell}_{r-1} y} \\ &= (q-1) R^{H,x}_{\ell_k y', \bar{\ell}_{r-1} y} + q R^{H,x}_{\bar{\ell}_{k+1} y', \bar{\ell}_{r-1} y} = (q-1) R^{H,x}_{u, M(w)} + q R^{H,x}_{M(u), M(w)}, \end{aligned}$$
where both the second and the fourth equalities hold by Corollary 3.6.
- $M(u) = \rho_s(c_k) y'$ and $M(u) \lhd u$. In this case $1 \leq k \leq r-1$. Hence
$$R^{H,x}_{u,w} = R^{H,x}_{c_k y', \ell_r y} = R^{H,x}_{\ell_k y', \ell_r y} = R^{H,x}_{\bar{\ell}_{k-1} y', \bar{\ell}_{r-1} y} = R^{H,x}_{c_{k-1} y', \bar{\ell}_{r-1} y} = R^{H,x}_{M(u), M(w)},$$
where both the second and the fourth equalities hold by Corollary 3.6.
- $M(u) = \rho_s(c_k) y'$ and $M(u) \rhd u$. In this case $0 \leq k \leq r-2$. Hence
$$\begin{aligned} R^{H,x}_{u,w} = R^{H,x}_{c_k y', \ell_r y} = R^{H,x}_{\bar{\ell}_k y', \ell_r y} &= (q-1) R^{H,x}_{\bar{\ell}_k y', \bar{\ell}_{r-1} y} + q R^{H,x}_{\ell_{k+1} y', \bar{\ell}_{r-1} y} \\ &= (q-1) R^{H,x}_{c_k y', \bar{\ell}_{r-1} y} + q R^{H,x}_{c_{k+1} y', \bar{\ell}_{r-1} y} = (q-1) R^{H,x}_{u, M(w)} + q R^{H,x}_{M(u), M(w)}, \end{aligned}$$

where both the second and the fourth equalities hold by Corollary 3.6.

Case (4) In this case $p = t$ and $s \leq y$, hence M commutes with ρ_s in $[e, w_0(s,t)]$ by R5. Thus we can proceed exactly as in the case (3), by substituting $r+1$ to r, since $L(w_0(s,t)) = r+1$.

Suppose now M be associated with a left system (J, s, t, M_{st}). If ${}_J w^{\{s\}} \neq e$ we can proceed as in the proof of [9, Theorem 4.5], otherwise $w = \ell_r y$, $s, t \notin D_L(y)$ (note that $n \in \{r, r+1\}$); moreover, we suppose $y \neq e$, since H-special matchings are calculating for dihedral Coxeter groups by Theorem 2.6.

Under these hypoteses, (J, s, t, M_{st}) is a left system for w if and only if L1, L2, L3 and L4 hold. Also, any element $u \leq w$ is of the form $c_k y'$, where $c \in \{s, t\}$, $s,\ t \notin D_L(y)$ and $y' \leq y$, therefore $c_k = ({}_J u)_{\{s\}} \cdot {}_{\{s,t\}}({}^J u)$ and $y' = {}^{\{s,t\}}\,({}^J u)$. Note that M_{st} does not commute with both ρ_s and ρ_p (hence, by L4, s and t are not both $\leq y$), otherwise M would coincide with λ_ℓ. The cases that can occur are the following ones:

(1) $\ell = s$, $p \not\leq y$ and M_{st} does not commute with $\rho_{\bar{p}}$;
(2) $\ell = s$, $p \leq y$;
(3) $M(\ell_n) = \ell_{n-1}$ and either $\ell = t$ or M_{st} commutes with $\rho_{\bar{p}}$;
(4) $\ell = t$, $M(\ell_n) = \ell_{n-1}$.

Case (1). The hypotheses imply $\bar{p} \not\leq y$, $w_0(s,t) = s_r$ and exclude that simultaneously $\bar{p} \in H$ and $p \notin H$ by Lemma 4.4. If **a)** holds, define M^1 exactly as in Case (1), **a)** of the right systems. Now, if M^1 does not coincide with λ_ℓ on s_r, then $C_{st} : M^0 \to M^1 \to \lambda_\ell$ is a calculating chain for s_r; otherwise, define M^2 so that:

- $M^2(s_r) \neq M^1(s_r)$, $M^2(s_r) \lhd s_r$ and $M^2(M^1(s_r)) = M^1(M^2(s_r))$;
- for elements u not yet considered, set $M^{j+1}(u) = M^j(u)$.

Now, $C_{st} : M^0 \to \ldots M^2 \to \lambda_s$ is a calculating chain for s_r. By construction, the defined matchings are H-special matching of s_r associated to left systems, which are also left systems for w. In fact, by the simplification of the axioms above mentioned, we need tp check only L1, L2, L3 and L4. These properties hold since $w_0(s,t) = s_r$, M is a left H-special matching of w and by the definition of the other matchings of C_{st}. Hence, for each matching M^i of C_{st}, it is possible to consider the associated matching of w, which is H-special by Theorem 2.3. Thus C_{st} can be extended to a calculating chain of w, and we are done by Proposition 4.3.

If **b)** holds, define M^1 exactly as in the Case (1), **b)** of the right systems. Now, the cardinality of the orbit $\langle M^1, M^0\rangle(s_r)$ is a multiple of $|\langle M^1, M^0\rangle(u)|$ for every $u < s_r$, $u \notin \langle M^1, M^0\rangle(s_r)$. Hence, if M^1 commutes with λ_ℓ, we have a weak calculating chain C_{st} for s_r; otherwise, we note that M^1 is an H-special matching for ℓ_r (associated to a left system), for which **a)** holds. Thus we can refer to the previous case to obtain a weak calculating chain C_{st} for $w_0(s,t)$. As before, C_{st} can be extended to a weak calculating chain for w, and we are done by Proposition 4.3.

Case (2). By the hypotheses, M^0 commutes with ρ_p by L4 and it does not commute with $\rho_{\bar{p}}$ (otherwise M would coincide with λ_ℓ), hence $\bar{p} \not\leq y$ and $w_0(s,t) = s_r$. If **a)** holds, let us proceed as in the above Case (1) for left systems. If **b)** holds, necessarily $M(s_r) = s_{r-1}$, $r \geq 6$ and t_{r-1} is locally maximal for (M, λ_t). Hence we can proceed as in Case (2), **b)** for the right systems, in order to obtain a weak calculating chain C_{st}

for s_r. By construction, each matching of C_{st} is an H-special matching associated to a left system of s_r, which is also a left system for w. This allows to extend C_{st} to a weak calculating chain for w, and we are done by Proposition 4.3.

Case (3). In this case, which includes $\bar{p} \leq y$, that is $\ell_r < w_0(s,t) = \ell_n$, we can proceed as in Case (3) of the special matchings associated to the right systems.

Case (4). The hypotheses imply $\bar{p} \not\leq y$ (otherwise it would be $M(w) \triangleright w$), hence $w_0(s,t) = \ell_r$. Let $u \leq w$, $u = c_k y'$, $k \leq r$, $c \in \{s,t\}$, $s,t \notin D_L(y')$. The following situations can occur:

- $M(u) = \lambda_s(u)$ and $M(u) \triangleleft u$. In this case $c = s$, $1 \leq k \leq r-1$. Hence

$$R^{H,x}_{u,w} = R^{H,x}_{s_k y', t_r y} = R^{H,x}_{s_k y', s_r y} = R^{H,x}_{t_{k-1} y', t_{r-1} y} = R^{H,x}_{M(u),M(w)}$$

  where the third equality holds by Lemma 3.7 .
- $M(u) = \lambda_s(u)$ and $M(u) \triangleright u$. Hence $c = t$, $0 \leq k < r-1$.

$$R^{H,x}_{u,w} = R^{H,x}_{t_k y', t_r y} = R^{H,x}_{t_k y', s_r y} = (q-1)R^{H,x}_{t_k y', t_{r-1} y} + qR^{H,x}_{s_{k+1} y', t_{r-1} y} = (q-1)R^{H,x}_{u,M(w)} + qR^{H,x}_{M(u),M(w)},$$

  where the second equality holds by Lemma 3.7.
- $M(u) = \lambda_t(u)$ and $M(u) \triangleleft u$. In this case $c = t$, $1 \leq k < r-1$. Hence

$$R^{H,x}_{u,w} = R^{H,x}_{t_k y', t_r y} = R^{H,x}_{s_{k-1} y', s_{r-1} y} = R^{H,x}_{s_{k-1} y', t_{r-1} y} = R^{H,x}_{M(u),M(w)},$$

  where the third equality holds by Lemma 3.7 .
- $M(u) = \lambda_t(u)$ and $M(u) \triangleright u$. In this case $c = s$, $0 \leq k < r-2$. Hence

$$R^{H,x}_{u,w} = R^{H,x}_{s_k y', t_r y} = (q-1)R^{H,x}_{s_k y', s_{r-1} y} + qR^{H,x}_{t_{k+1} y', s_{r-1} y} = (q-1)R^{H,x}_{s_k y', t_{r-1} y} + q)R^{H,x}_{t_{k+1} y', t_{r-1} y} = (q-1)R^{H,x}_{u,M(w)} + qR^{H,x}_{M(u),M(w)},$$

  where the third equality holds by Lemma 3.7.
- $M(u) = \rho_q(c_k)y'$ and $M(u) \triangleleft u$, for q generator in $\{s,t\}$. In this case $0 < k \leq r$. If $k = r$ (resp. $k = r-1$), which implies $c = t$ (resp. $c = s$), trivially M calculates $R^{H,x}_{u,w}$; otherwise

$$R^{H,x}_{u,w} = R^{H,x}_{t_k y', t_r y} = R^{H,x}_{s_{k-1} y', s_{r-1} y} = R^{H,x}_{s_{k-1} y', t_{r-1} y} = R^{H,x}_{t_{k-1} y', t_{r-1} y} = R^{H,x}_{M(u),M(w)},$$

  where both the first and the fourth equalities holds by Corollary 3.6 and the third equality holds by Lemma 3.7.
- $M(u) = \rho_q(c_k)y'$ and $M(u) \triangleright u$, for q suitable element in $\{s,t\}$. In this case $0 \leq k \leq r-1$. If $k = r-1$, hence $c = t$ and trivially M calculates $R^{H,x}_{u,w}$; otherwise

$$R^{H,x}_{u,w} = R^{H,x}_{s_k y', t_r y} = (q-1)R^{H,x}_{s_k y', s_{r-1} y} + qR^{H,x}_{t_{k+1} y', s_{r-1} y} = (q-1)R^{H,x}_{c_k y', s_{r-1} y} + qR^{H,x}_{c_{k+1} y', s_{r-1} y} = (q-1)R^{H,x}_{c_k y', t_{r-1} y} + qR^{H,x}_{c_{k+1} y', t_{r-1} y} = (q-1)R^{H,x}_{u,M(w)} + qR^{H,x}_{M(u),M(w)},$$

  where both the first and the third equalities holds by Corollary 3.6 and the fourth equality hold by Lemma 3.7.

So the proof is complete. □

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Dipartimento di Ingegneria Industriale e Scienze Matematiche, Università Politecnica delle Marche, Via Brecce Bianche, 60131 Ancona, Italy

E-mail address: `telloni@dipmat.univpm.it`