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A priori estimates for solutions of p–Kirchhoff systems under dynamic boundary conditions

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Dedicated to Patrizia Pucci on the occasion of her 60th birthday with feelings of great admiration and gratitude

Abstract. In this paper we consider perturbed evolution systems governed by the p–Kirchhoff operator in bounded domains. These models are characterized by time dependent nonlinear driving forces and boundary damping terms. The question of non–continuation of maximal solutions is treated and some a priori estimates for the lifespan of solutions are given.

1. Introduction

In this paper we are interested in p–Kirchhoff systems involving nonlinear driving and damping terms in bounded domains, under dynamic boundary conditions. More precisely, we study the problem

$$\begin{cases} u_{tt} - M\left(\|Du(t,\cdot)\|_p^p\right)\Delta_p u + \mu|u|^{p-2}u = f(t,x,u), & \text{in } \mathbb{R}_0^+ \times \Omega, \\ u(t,x) = 0, & \text{on } \mathbb{R}_0^+ \times \Gamma_0, \\ u_{tt} = -\left[M(\|Du(t,\cdot)\|_p^p)|Du|^{p-2}\partial_\nu u + Q(t,x,u,u_t)\right], & \text{on } \mathbb{R}_0^+ \times \Gamma_1. \end{cases} \tag{1.1}$$

The function $u = (u_1, \dots, u_N) = u(t,x)$ represents the vectorial displacement, $N \ge 1$, $\mathbb{R}_0^+ = [0,\infty)$ and Ω is a regular bounded domain of $\mathbb{R}^n$. We assume that $\partial\Omega = \Gamma_0 \cup \Gamma_1$, $\Gamma_0 \cap \Gamma_1 = \emptyset$ and $\mu_{n-1}(\Gamma_0) > 0$, where μ_{n-1} denotes the $(n-1)$–dimensional Lebesgue measure on $\partial\Omega$. The exponent $p > 1$ and $\Delta_p u = \operatorname{div}(|Du|^{p-2}Du) = \operatorname{div}(|Du|^{p-2}Du_1, \dots, |Du|^{p-2}Du_N)$ is the vectorial p–Laplacian operator.

The *Kirchhoff dissipative function* M is assumed of the standard form

$$M(\tau) = a + b\gamma\tau^{\gamma-1}, \quad a,b \ge 0, \quad a+b > 0, \tag{1.2}$$

with $\gamma > 1$ if $b > 0$, and $\gamma = 1$ if $b = 0$. Problem (1.1) is said to be *non–degenerate* when $a > 0$, otherwise (1.1) is called *degenerate*. The term $\mu|u|^{p-2}u$, with $\mu \ge 0$, is a nonlinear perturbation acting on the system.

Following [**22**], *we take the internal nonlinear source force* f *of the type*

$$f(t,x,u) = g(t,x)|u|^{\sigma-2}u + c(x)|u|^{q-2}u, \tag{1.3}$$

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where $1 \leq \sigma < q$*, the function* $c \in L^\infty(\Omega)$ *is non–negative,* $g \in C(\mathbb{R}_0^+ \times \Omega)$ *is non–positive, differentiable with respect to* t *and* $g_t \in C(\mathbb{R}_0^+ \times \Omega)$. More specific assumptions on f will be given in Section 2. The *negative term of* $(f(t,x,u),u)$, deriving from g, makes the analysis more delicate than in [**23**, **24**], since it *works against the blow up* and against the non–continuation of local solutions. ¿From here on $(\cdot,\cdot)$ denotes the usual scalar product in $\mathbb{R}^N$.

Concerning the *external nonlinear boundary damping* Q*, we suppose that*

$$Q(t,x,u,v) = d_1(t,x)|u|^\kappa |v|^{m-2}v + d_2(t,x,u)|v|^{\wp-2}v,$$

where d_1 *and* d_2 *are non–negative continuous functions, satisfying integrability conditions with respect to the space variable, and* κ*,* m*,* $\wp$ *are positive constants such that* $\kappa \geq 0$ *and* $1 < m \leq \wp - \kappa$. More detailed assumptions on Q will be stated in Section 2.

The interest in p–Kirchhoff models, besides the mathematical curiosity, derives from the several applications they have in reaction–diffusion theory and in non–Newtonian theory, where it is evident the role of each term of the system in the global behavior of the body. For example, thinking of fluids, the quantity p is characteristic of the medium, and its magnitude is representative of the elastic and/or pseudoplastic properties of the fluid, see [**3**, **19**] and the references therein.

The boundary conditions in (1.1) express the fact that the system does not neglect acceleration terms on the boundary. They are usually called *dynamic boundary conditions* and arise in several physical applications. In one dimension and in the scalar case, problem (1.1) models the dynamic evolution of a viscoelastic rod fixed at one end and with a tip mass attached to its free end. The dynamic boundary conditions represent the Newton law for the attached mass, cfr. [**2**, **11**, **18**]. In the two dimensional space and for $N = 1$, these boundary conditions appear in the transverse motions of a flexible membrane Ω which boundary $\partial\Omega$ may be affected by vibrations only in the region Γ_1, see [**17**]. More details on the physical meaning of the boundary conditions in (1.1), as well as on the so–called *acoustic boundary conditions* for exterior domains in $\mathbb{R}^3$, can be found in [**6**, **9**, **13**].

In the last years there has been an increasing attention towards problems involving dynamic boundary conditions, and many different related topics have been considered. In [**20**] the author studies the well–posedness of initial–boundary value wave problems and the qualitative properties of the solutions. For the existence and asymptotic stability of solutions of strongly damped wave equations, even with delay terms, we quote [**16**] and the references therein. The recent paper [**15**], somehow based on [**24**], treats the blow up of solutions of the strongly damped model

$$\begin{cases} u_{tt} - \Delta u - \varrho\Delta u_t = |u|^{q-2}u, & \text{in } \mathbb{R}_0^+ \times \Omega, \\ u(t,x) = 0, & \text{on } \mathbb{R}_0^+ \times \Gamma_0, \\ u_{tt} = -\left[\partial_\nu u + \varrho\partial_\nu u_t + ru_t\right], & \text{on } \mathbb{R}_0^+ \times \Gamma_1, \end{cases} \tag{1.4}$$

with $\varrho > 0$, $r > 0$ and $q > 2$. The exponential growth at infinity of the energy has been analyzed in [**14**] for the nonlinear damping case, that is when ru_t in (1.4) is replaced by $r|u_t|^{m-2}u_t$, with $m \geq 2$. The energy estimates given in [**14**] have been extended in [**5**] to the more general system

$$\begin{cases} u_{tt} - M\left(\|Du(t,\cdot)\|_2^2\right)\Delta u - \varrho(t)\Delta u_t + \mu u = f(t,x,u), & \text{in } \mathbb{R}_0^+ \times \Omega, \\ u(t,x) = 0, & \text{on } \mathbb{R}_0^+ \times \Gamma_0, \\ u_{tt} = -\big[M(\|Du(t,\cdot)\|_2^2)\partial_\nu u + \varrho(t)\partial_\nu u_t + Q(t,x,u,u_t)\big], & \text{on } \mathbb{R}_0^+ \times \Gamma_1, \end{cases}$$

where $\varrho \in C(\mathbb{R}_0^+)$ is a nonnegative function.

The present paper is connected with [**4**, **6**]. In [**4**] we give a priori estimates for the *lifespan* T of maximal solutions of polyharmonic Kirchhoff systems, under homogeneous Dirichlet boundary conditions. The lifespan T of a solution u is defined by

$$T = \sup\{t > 0 : u \text{ exists in } [0,t)\}.$$

In [**6**] we treat the question of global non–existence of solutions of (1.1), and here we complete the picture, obtaining lifespan estimates for them.

The main result of this paper is Theorem 3.1, in which an upper bound T_0 for T is found, when the initial data belong to an appropriate region Σ_0 in the phase plane. Indeed, we identify two critical values E_0 and υ_0, with the property that if $Eu(0) < E_0$ and $\|Du(0,\cdot)\|_p > \upsilon_0$, then $T \le T_0$. Here $Eu(0)$ and $\|Du(0,\cdot)\|_p$ are the energy of the system along a solution u and the Sobolev norm of u at the time zero, respectively. Moreover, T_0 depends only on the initial data and on the parameters of (1.1).

Theorem 3.1 extends Theorem 6.1 of [**4**] to the case of p–Kirchhoff systems with dynamic boundary conditions. The key points in the proof are Sobolev type embeddings given in [**10**] and a deep use of the energy functional E associated to (1.1). The study of the geometric features of the model, connected with the properties of E, leads to a crucial qualitative analysis of the problem. In particular, Lemma 2.2 and Proposition 2.4 are essential in the proof of the non–continuation Theorem 3.1.

The extension of Theorem 6.1 of [**4**] to (1.1) presents several difficulties. Indeed, as in [**6**], the boundary action of Q forces the choice of a new functional setting, together with Sobolev interpolation embeddings, and requires additional global lower bounds for $\|Du(t,\cdot)\|_p$ in the energy estimates. Consequently, the expression of T_0 given in Theorem 3.1 is much more involved than the corresponding value obtained in [**4**] for polyharmonic Kirchhoff systems.

The delicate argument of the proof of Theorem 3.1 guarantees global non–existence of solutions of (1.1), but it does not establish by itself that maximal solutions blow up at the lifespan T. It is worth noting that in general the proofs of global non–existence in the literature do not imply finite time blow up of the solutions. Indeed, without a local continuation argument, the solution, before becoming unbounded, could leave the domain of one of the differential operators involved in the problem. For a more detailed discussion on this point we refer the interested reader to [**4**, **8**] and the references therein.

In Corollary 3.3 we obtain a finite time blow up result, extending Corollary 6.2 of [**4**] to (1.1). Finally, in Corollary 3.4 we give simplified expressions of T_0, when Q is of a special form interesting in applications. As far as we know, this paper is the first attempt to give lifespan estimates for maximal solutions of p–Kirchhoff systems governed by nonlinear driving and dissipative boundary forces.

2. Preliminaries

The functional setting. For simplicity consider $1 < p < n$, and denote by $L^p(\Omega)$ the usual Lebesgue space equipped with the norm $\|\phi\|_p = \left(\int_\Omega |\phi(x)|^p\,dx\right)^{1/p}$. If $\omega \geq 1$, we endow $L^\omega(\Gamma_1)$ with the norm $\|\phi\|_{\omega,\Gamma_1} = \left(\int_{\Gamma_1} |\phi(x)|^\omega d\mu_{n-1}\right)^{1/\omega}$. Let

$$W^{1,p}_{\Gamma_0}(\Omega) = \{\phi \in W^{1,p}(\Omega) \,:\, \phi|_{\Gamma_0} = 0\},$$

equipped with the norm $\|\phi\|_{W^{1,p}_{\Gamma_0}(\Omega)} = \|D\phi\|_p$, where $\phi|_{\Gamma_0} = 0$ is understood in the trace sense. In the following, we shall simply denote $\|\phi\|_{W^{1,p}_{\Gamma_0}(\Omega)}$ by $\|\phi\|$. The norm $\|\cdot\|$ is equivalent to $\|\cdot\|_{W^{1,p}(\Omega)}$ by the Poincaré inequality, see [**25**, Corollary 4.5.3 and Theorem 2.6.16]. In particular, inequality (4.5.2) of [**25**] reduces to

$$\|\phi\|_{p^*} \leq \mathfrak{C}_{p^*}\|D\phi\|_p \quad \text{for all } \phi \in W^{1,p}_{\Gamma_0}(\Omega), \tag{2.1}$$

where $p^* = np/(n-p)$, $\mathfrak{C}_{p^*} = C(n, N, p, \Omega)\cdot[B_{1,p}(\Gamma_0)]^{-1/p}$, and the Bessel capacity $B_{1,p}(\Gamma_0) > 0$ since $\mu_{n-1}(\Gamma_0) > 0$, cf. [**25**, Theorem 2.6.16]. Then, the embedding $W^{1,p}_{\Gamma_0}(\Omega) \hookrightarrow L^q(\Omega)$ is continuous whenever $1 \leq q \leq p^*$, and so there exists a constant $\mathfrak{C}_q > 0$ such that

$$\|\phi\|_q \leq \mathfrak{C}_q\|D\phi\|_p \quad \text{for all } \phi \in W^{1,p}_{\Gamma_0}(\Omega). \tag{2.2}$$

Similarly, for $s \in (0, 1)$, let

$$W^{s,p}_{\Gamma_0}(\Omega) = \{\phi \in W^{s,p}(\Omega) \,:\, \phi|_{\Gamma_0} = 0\},$$

equipped with the norm $\|\phi\|_{W^{s,p}_{\Gamma_0}(\Omega)} = \|\phi\|_{W^{s,p}(\Omega)}$, where $W^{s,p}(\Omega)$ is the fractional Sobolev space of order s, see [**1**].

The elementary bracket pairing $\langle\varphi, \psi\rangle = \int_\Omega(\varphi(x), \psi(x))dx$ is clearly well defined for all φ, ψ such that $(\varphi, \psi) \in L^1(\Omega)$ and $\langle u, \phi\rangle_{\Gamma_1} = \int_{\Gamma_1}(u(x), \phi(x))d\mu_{n-1}$ is well defined for all u, ϕ such that $(u, \phi) \in L^1(\Gamma_1)$.

Since we are in the vectorial setting, for simplicity we shall use the notation $L^p(\Omega)$ also to denote the product space $[L^p(\Omega)]^N$ or $[L^p(\Omega)]^{nN}$, and the same agreement will be adopted for all the other spaces involved in the treatment.

The set

$$X = C(I \to W^{1,p}_{\Gamma_0}(\Omega)) \cap C^1(I \to L^2(\Omega))$$

is *the solution and test function space.* Here $I = [0, T)$, with $T \in (0, \infty]$, is the maximal time existence interval for a solution $u \in X$ of (1.1). In other words, the *lifespan* T of u is defined by

$$T = \sup\{t > 0 \,:\, u \text{ exists in } [0, t)\}.$$

In what follows $p_* = p(n-1)/(n-p)$ and $(p_n)_{n=1}^\infty$, with

$$\frac{2n}{n+1} < p_n = \frac{1}{2}[\sqrt{(n+1)^2 + 4n} + 1 - n] < 2. \tag{2.3}$$

Clearly, $(p_n)_{n=1}^\infty$ is a strictly increasing sequence, with $p_3 \doteq 1,65$ and $\lim_{n\to\infty} p_n = 2$, cfr. [**6**].

Proposition 2.1 (Proposition 3.1 of [**6**]). *Given* $p \in (p_n, n)$ *and* $q > \max\{2, p\}$, *then*

$$\wp_0 = \frac{pq(n-1+p) - p^2(n-1)}{n(q-p) + p^2} \in (\max\{2, p\}, \min\{p_*, q\}). \tag{2.4}$$

From the proof of Proposition 2.1 it is clear that the assumption $p > p_n$ is needed only to show that $\wp_0 > 2$. Condition $\wp_0 > 2$ is crucial in the proof of the main Theorem 3.1. Of course, if $1 < p < n$, in order to have $p_n < n$ it is enough to take $n > 3/2$, that is $n \geq 2$.

On the internal source force f and the external damping Q. We are going to present some prototypes for f and Q, first introduced in [**22**], in the form given in [**4**, **6**].

The *nonlinear term* $f : \mathbb{R}_0^+ \times \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ *in* (1.3) *satisfies the further assumption*

$$\begin{gathered} 1 \leq \sigma < q, \quad \max\{2, \gamma p\} < q \leq p^*, \quad c_\infty = \|c\|_\infty > 0, \quad \overline{c} = \operatorname{ess\,inf}_\Omega c > 0; \\ 0 \leq -g(t,x),\ g_t(t,x) \leq h(x) \ \textit{in}\ \mathbb{R}_0^+ \times \Omega,\ \textit{for some}\ h \in L^1(\Omega), \\ g(t,\cdot) \in L^{q/(q-\sigma)}(\Omega) \ \textit{in}\ \mathbb{R}_0^+. \end{gathered} \tag{2.5}$$

Clearly, *condition* $\max\{2, \gamma p\} < q \leq p^*$ *implies* $1 \leq \gamma < n/(n-p)$ *and* $p > 2n/(n+2)$.

Thanks to the integrability properties of g and to the boundedness of c, it results that $(f(t,x,\phi), \phi(t,x)) \in L^1(\Omega)$ for all $t \in \mathbb{R}_0^+$ and for all $\phi \in W^{1,p}_{\Gamma_0}(\Omega)$.

Moreover, as shown in [**6**, Lemma 4.1] (see also [**7**, Lemma 4.1]), the function f admits a *potential* $F : \mathbb{R}_0^+ \times \Omega \times \mathbb{R}^N \to \mathbb{R}$, that is $f(t,x,\phi) = \nabla_\phi F(t,x,\phi)$, with $F(t,x,0) = 0$ and

$$F(t,x,\phi) = g(t,x)\frac{|\phi|^\sigma}{\sigma} + c(x)\frac{|\phi|^q}{q}.$$

Of course, for any $(t,x,\phi) \in \mathbb{R}_0^+ \times \Omega \times W^{1,p}_{\Gamma_0}(\Omega)$, the potential F is well defined and of class $L^1(\Omega)$. In other words,

$$\mathscr{F}\phi(t) = \mathscr{F}(t,\phi) = \int_\Omega \left\{ g(t,x)\frac{|\phi(t,x)|^\sigma}{\sigma} + c(x)\frac{|\phi(t,x)|^q}{q} \right\} dx \tag{2.6}$$

for all $\phi \in W^{1,p}_{\Gamma_0}(\Omega)$. Thus, differentiation under the integral sign gives

$$\mathscr{F}_t(t,\phi) = \int_\Omega g_t(t,x)\,\frac{|\phi(x)|^\sigma}{\sigma}\,dx \geq 0 \quad \text{for all } (t,\phi) \in \mathbb{R}_0^+ \times W^{1,p}_{\Gamma_0}(\Omega).$$

Finally, $\langle f(t,\cdot,\phi), \phi(t,\cdot)\rangle \in L^1_{\mathrm{loc}}(\mathbb{R}_0^+)$ along any $\phi \in W^{1,p}_{\Gamma_0}(\Omega)$ and

$$q\mathscr{F}\phi(t) \leq \langle f(t,x,\phi(t,x)), \phi(t,x)\rangle \leq c_\infty \|\phi(t,\cdot)\|_q^q, \tag{2.7}$$

for all $t \in \mathbb{R}_0^+$ and $\phi \in W^{1,p}_{\Gamma_0}(\Omega)$, being $\sigma < q$ and $g \leq 0$.

Concerning the boundary damping Q, *assume that for all* $(t,x,u,v) \in \mathbb{R}_0^+ \times \Gamma_1 \times \mathbb{R}^N \times \mathbb{R}^N$

$$\begin{gathered} Q(t,x,u,v) = d_1(t,x)|u|^\kappa |v|^{m-2}v + d_2(t,x,u)|v|^{\wp-2}v, \\ 1 < m \leq \wp - \kappa, \quad 0 \leq \kappa \leq p(1 - m/\wp), \quad 2 \leq \wp < \wp_0, \end{gathered} \tag{2.8}$$

where $d_1 \in C(\mathbb{R}_0^+ \to L^{\wp_1}(\Gamma_1))$ *and* $d_2 \in C(\mathbb{R}_0^+ \to L^\infty(\Gamma_1))$ *are non–negative and*

$$\wp_1 = \begin{cases} \wp/(\wp - \kappa - m), & \text{if } \wp > m + \kappa, \\ \infty, & \text{if } \wp = m + \kappa. \end{cases}$$

Usually in the literature the dissipative function Q is considered in the simplified form in which $d_1(t,x) \equiv d_1 > 0$, $\kappa = 0$ and $d_2 \equiv 0$.

The energy of the system. For all $\tau \in \mathbb{R}_0^+$ we set $\mathscr{M}(\tau) = a\tau + b\tau^\gamma$, so that

$$\gamma \mathscr{M}(\tau) \geq \tau M(\tau) \quad \text{for all } \tau \in \mathbb{R}_0^+. \tag{2.9}$$

The *total energy of the field* $\phi \in X$ associated with (1.1) is

$$E\phi(t) = \frac{1}{2}(\|\phi_t(t,\cdot)\|_2^2 + \|\phi_t(t,\cdot)\|_{2,\Gamma_1}^2) + \mathscr{A}\phi(t) - \mathscr{F}\phi(t), \tag{2.10}$$

where $\mathscr{F}$ is given in (2.6) and

$$p\mathscr{A}\phi(t) = \mathscr{M}(\|D\phi(t,\cdot)\|_p^p) + \mu\|\phi(t,\cdot)\|_p^p \geq 0,$$

by (1.2), being $\mu \geq 0$. Of course $E\phi$ is well defined in X by (2.5).

For all $\phi \in X$ and $(t,x) \in \mathbb{R}_0^+ \times \Omega$ put pointwise

$$A\phi(t,x) = -M(\|D\phi(t,\cdot)\|_p^p)\Delta_p\phi(t,x) + \mu|\phi(t,x)|^{p-2}\phi(t,x),$$

so that A is the Fréchet derivative of $\mathscr{A}$ with respect to ϕ, and

$$\begin{aligned} \langle\langle A\phi(t,\cdot), \phi(t,\cdot)\rangle\rangle : &= \langle A\phi(t,\cdot), \phi(t,\cdot)\rangle_{(W^{1,p}_{\Gamma_0}(\Omega), [W^{1,p}_{\Gamma_0}(\Omega)]')} \\ &= M(\|D\phi(t,\cdot)\|_p^p)\|D\phi(t,\cdot)\|_p^p + \mu\|\phi(t,\cdot)\|_p^p \\ &\leq \gamma p\mathscr{A}\phi(t), \end{aligned} \tag{2.11}$$

by (1.2), (2.9), being $\mu \geq 0$ and $\gamma \geq 1$.

Following [**6**, **21**], we say that $u \in X$ is a (*weak*) *solution of* (1.1) if u satisfies:

(A) *Distribution Identity*

$$\langle u_t, \phi\rangle\big]_0^t = \int_0^t \Big\{\langle u_t, \phi_t\rangle - M(\|Du(\tau,\cdot)\|_p^p)\cdot\langle|Du|^{p-2}Du, D\phi\rangle - \mu\langle|u|^{p-2}u, \phi\rangle + \langle f(\tau,\cdot,u), \phi\rangle - \langle Q(\tau,\cdot,u,u_t) + u_{tt}, \phi\rangle_{\Gamma_1}\Big\}d\tau$$

for all $t \in I$ and $\phi \in X$;

(B) *Energy Conservation*

$$(i) \qquad \mathscr{D}u(t) = \langle Q(t,\cdot,u(t,\cdot),u_t(t,\cdot)), u_t(t,\cdot)\rangle_{\Gamma_1} + \mathscr{F}_t u(t) \in L^1_{\mathrm{loc}}(I),$$

$$(ii) \qquad Eu(t) \leq Eu(0) - \int_0^t \mathscr{D}u(\tau)d\tau \quad \text{for all } t \in I.$$

Observe that $\mathscr{D}u \geq 0$ in I, being $\mathscr{F}_t u \geq 0$ and $\langle Q(t,\cdot,u(t,\cdot),u_t(t,\cdot)), u_t(t,\cdot)\rangle_{\Gamma_1} \geq 0$ by (1.3), (2.5) and (2.8).

To make the *Distribution Identity* meaningful *we assume that* $\langle Q(t,\cdot,u,u_t), \phi\rangle_{\Gamma_1}$ *and* $\langle u_{tt}, \phi\rangle_{\Gamma_1}$ *are in* $\in L^1_{\mathrm{loc}}(I)$, *along any field* $\phi \in X$. The other terms in the *Distribution Identity* (A) are well defined thanks to the choice of f, Q and X.

Some auxiliary results. From here on, we put $\varsigma = a$ if $b = 0$ or $\varsigma = b$ if $b > 0$ in (1.2). Moreover, *if* $u \in X$ *is a solution of* (1.1) *we shall write* $\upsilon(t) = \|Du(t,\cdot)\|_p$ *for each* $t \in I$.

Lemma 2.2 (Lemma 4.4 of [**6**]). *Assume* (1.3) *and* (2.5). *If* $u \in X$ *is a solution of* (1.1), *then for all* $t \in I$

$$Eu(t) \geq \varphi(\upsilon(t)) = \frac{\varsigma}{p}\upsilon(t)^{\gamma p} - \frac{c}{q}\upsilon(t)^q, \tag{2.12}$$

where $c = c_\infty\mathfrak{C}_q^q$ *and* $\mathfrak{C}_q$ *is the embedding constant introduced in* (2.2).

The function $\varphi : \mathbb{R}_0^+ \to \mathbb{R}$ introduced in Lemma 2.2 attains its maximum at

$$\upsilon_0 = \left(\frac{\varsigma\gamma}{c}\right)^{1/(q-\gamma p)}.$$

Moreover, φ is strictly decreasing for $\upsilon \geq \upsilon_0$, with $\varphi(\upsilon) \to -\infty$ as $\upsilon \to \infty$. Finally,

$$\varphi(\upsilon_0) = \left(1 - \frac{\gamma p}{q}\right) w_0 = E_0 > 0, \quad \text{where} \quad w_0 = \frac{\varsigma\upsilon_0^{\gamma p}}{p} > 0, \\ \Sigma_0 = \{(\upsilon, E) \in \mathbb{R}^2 \,:\, \upsilon > \upsilon_0,\ E < E_0\}. \tag{2.13}$$

In the sequel, given a solution $u \in X$ of (1.1), we put for convenience

$$w_1 = \inf_{t\in I} \mathscr{A}u(t), \quad w_2 = \inf_{t\in I} \mathscr{F}u(t), \\ E_1 = \left(1 - \frac{\gamma p}{q}\right) w_1, \quad E_2 = \left(\frac{q}{\gamma p} - 1\right) w_2. \tag{2.14}$$

The next lemma establishes some crucial properties, deriving from the geometry of the system, which link the energy functional E to the main elliptic part $\mathscr{A}$ and the potential $\mathscr{F}$. For polyharmonic Kirchhoff systems with internal damping, a similar result has been proved in [**4**]. The main steps are formally the same, but for the sake of clarity and completeness, we write them below, since the functional $\mathscr{A}$ is essentially different from the corresponding elliptic functional of [**4**]. In the stationary case and for higher order models, we refer to [**12**] for the existence of solutions of p–polyharmonic Kirchhoff systems under homogeneous Dirichlet boundary conditions.

From now on, $u \in X$ is a fixed solution of (1.1) *such that $Eu(0) < E_0$.*

Lemma 2.3. *It results that $\upsilon_0 \notin \overline{\upsilon(I)}$ and $w_1 \neq w_0$. Moreover, the following are equivalent:*

(*i*) $w_1 > w_0$;
(*ii*) $\overline{\upsilon(I)} \subset (\upsilon_0, \infty)$;
(*iii*) $w_2 > \gamma p w_0/q$.

Finally, if one of the conditions (*i*)*–*(*iii*) *holds, then $E_0 < E_1 < E_2$.*

In particular, if $(\upsilon(0), Eu(0)) \in \Sigma_0$, then $(\upsilon(t), Eu(t)) \in \Sigma_0$ for all $t \in I$, properties (*i*)*–*(*iii*) *hold, $E_0 < E_1 < E_2$ and $w_2 > \gamma p w_1/q > \gamma p w_0/q$.*

Proof. Let $u \in X$ be a solution of (1.1) and assume that $Eu(0) < E_0$. Suppose by contradiction that $\upsilon_0 \in \overline{\upsilon(I)}$. Then there exists a sequence $(t_j)_j \subset I$ such that $\upsilon(t_j) \to \upsilon_0$ as $j \to \infty$. Now, by (2.12) we have $E_0 > Eu(0) \geq Eu(t_j) \geq \varphi(\upsilon(t_j))$, which provides $E_0 > E_0$ by the continuity of $\varphi \circ \upsilon$. This contradiction proves the claim.

We show that $w_1 \neq w_0$. Otherwise, $\mathscr{A}u(t) \geq w_1 = w_0$ for all $t \in I$. Therefore, by (2.7), (2.10) and (2.13), we have

$$\mathscr{A}u(t) - \frac{\varsigma\gamma}{q}\upsilon(t)^{\gamma p} \geq \left(1 - \frac{\gamma p}{q}\right)\mathscr{A}u(t) \geq E_1 = E_0 > Eu(0) \\ \geq \mathscr{A}u(t) - \frac{c}{q}\upsilon(t)^q, \tag{2.15}$$

so that $\upsilon(t) > \upsilon_0$ for each $t \in I$. Consequently, $\overline{\upsilon(I)} \subset (\upsilon_0, \infty)$. On the other hand, there exists a sequence $(t_j)_j$ such that $\mathscr{A}u(t_j) \to w_1 = w_0$ as $j \to \infty$, so that

$$\limsup_{j\to\infty} \upsilon(t_j) \le \lim_{j\to\infty} [p\mathscr{A}u(t_j)/\varsigma]^{1/\gamma p} = [pw_0/\varsigma]^{1/\gamma p} = \upsilon_0,$$

which contradicts the fact that $\overline{\upsilon(I)} \subset (\upsilon_0, \infty)$. Hence $w_1 \neq w_0$.

It remains to prove the equivalence of (i)–(iii).

$(i) \Rightarrow (ii)$. It is enough to show that $\upsilon(I) \subset (\upsilon_0, \infty)$, which immediately gives $\overline{\upsilon(I)} \subset (\upsilon_0, \infty)$, being $\upsilon_0 \notin \overline{\upsilon(I)}$. Relation $w_1 > w_0$ implies $E_1 > Eu(0)$. Then, repeating the calculation made in (2.15), we obtain again $\upsilon(t) > \upsilon_0$ for all $t \in I$.

$(ii) \Rightarrow (iii)$. If $\upsilon(t) > \upsilon_0$ for all $t \in I$, then $\mathscr{F}u(t) \ge w_0 - Eu(0) > w_0 - E_0 = \gamma p w_0/q$ for all $t \in I$ by (2.10) and so $w_2 > \gamma p w_0/q$.

$(iii) \Rightarrow (i)$. By (2.7) and (iii) we have

$$\frac{c}{q}\upsilon(t)^q \ge \mathscr{F}u(t) \ge w_2 > \frac{\gamma p}{q} w_0 = \frac{c}{q}\upsilon_0^q,$$

which implies $\upsilon(t) > \upsilon_0$ for all $t \in I$. Hence, $w_1 \ge w_0$ by (2.14). Consequently, we get $w_1 > w_0$, since the case $w_1 = w_0$ cannot occur.

Finally, if one of the conditions (i)–(iii) holds, then $E_0 < E_1$ by (i). Furthermore, $\mathscr{F}u(t) \ge w_1 - Eu(0) > w_1 - E_1 = \gamma p w_1/q$ for all $t \in I$ by (2.10). Hence, $w_2 > \gamma p w_1/q$ and so $E_1 < E_2$. In conclusion $E_0 < E_1 < E_2$, as claimed. The last part of the lemma follows at once from the previous arguments. □

The positive numbers introduced in (2.14) clearly depend on the fixed solution u of (1.1). Therefore, it is not possible to evaluate them. However, they play a crucial role in the next proposition, where a priori estimates on the Sobolev norm of u are obtained, see also[**4**]–[**7**] and [**21**]–[**23**]. These estimates are essential in the proof of Theorem 3.1.

Proposition 2.4. *For all $t \in I$*

$$\|u(t,\cdot)\|_q \ge c_1 \quad \text{and} \quad \|Du(t,\cdot)\|_p \ge c_1/\mathfrak{C}_q, \tag{2.16}$$

where $c_1 = (\gamma p w_0/c_\infty)^{1/q} > 0$ and $\mathfrak{C}_q$ is the Sobolev constant given in (2.2).

Furthermore, for all $t \in I$

$$p\mathscr{A}u(t) \ge a_1 \|Du(t,\cdot)\|_p^p, \tag{2.17}$$

where $a_1 = a + b(c_1/\mathfrak{C}_q)^{p(\gamma-1)} > 0$.

Proof. Let $u \in X$ be a solution of (1.1) as in the statement. By (2.7) and Lemma 2.3–(iii) we have that for all $t \in I$

$$\|u(t,\cdot)\|_q^q \ge \frac{q}{c_\infty}\mathscr{F}u(t) \ge \frac{q}{c_\infty} w_2 > \frac{\gamma p}{c_\infty} w_0,$$

which gives $(2.16)_1$. Hence, $(2.16)_2$ is true by (2.2). Finally, (2.17) is exactly formula (2.8) of [**6**]. □

Without loss of generality in what follows we assume that

$$c_1,\ c_1\mathfrak{C}_q^{-1} \in (0,1]. \tag{2.18}$$

3. Lifespan estimates for (1.1)

In Theorem 3.1 we give a priori estimates for the lifespan T of the maximal solutions of (1.1). We first list the structural assumptions on f, Q and the parameters of the problem. Then, we recall some Sobolev type inequalities, useful in the proofs. Finally, we state and prove Theorem 3.1, and give some corollaries interesting in applications.

Throughout the section, unless otherwise specified, take $p \in (p_n, n)$, *with* p_n *given in* (2.3). *Let* $\wp_0$ *be the positive number defined in* (2.4). *Assume* (1.3), (2.5), (2.8) *and define*

$$(3.1) \qquad \delta_1(t) = \|d_1(t,\cdot)\|_{\wp_1,\Gamma_1} \text{ and } \delta_2(t) = \sup_{(x,\xi)\in\Gamma_1\times\mathbb{R}^N} d_2(t,x,\xi) \quad \text{for all } t \in \mathbb{R}_0^+.$$

Since $\wp < \wp_0$ by (2.8), the embedding $L^{\wp_0}(\Gamma_1) \hookrightarrow L^{\wp}(\Gamma_1)$ is continuous and there exists $S_0 > 0$, such that $\|\phi\|_{\wp,\Gamma_1} \le S_0\|\phi\|_{\wp_0,\Gamma_1}$ for all $\phi \in L^{\wp_0}(\Gamma_1)$. The crucial parameter

$$s = \frac{n}{p} - \frac{n-1}{\wp_0} \in (0,1).$$

The embedding $W^{s,p}_{\Gamma_0}(\Omega) \hookrightarrow L^{\wp_0}(\Gamma_1)$ is continuous, thanks to [**1**, Theorem 7.58, with $\chi = 0$, $k = n-1$], being $\wp_0 > p$ by Proposition 2.1. In particular, there exists $S_1 > 0$ such that

$$(3.2) \qquad \|\phi\|_{\wp_0,\Gamma_1} \le S_1\|\phi\|_{W^{s,p}_{\Gamma_0}(\Omega)} \quad \text{for all } \phi \in W^{s,p}_{\Gamma_0}(\Omega).$$

Finally, by [**10**, Corollary 3.2–(a), with $s_1 = 0$, $s_2 = 1$, $p_1 = p_2 = p$ and $\theta = 1-s$], also the embedding $W^{1,p}_{\Gamma_0}(\Omega) \hookrightarrow W^{s,p}_{\Gamma_0}(\Omega)$ is continuous and so there exists $S_2 > 0$ such that for all $\phi \in W^{1,p}_{\Gamma_0}(\Omega)$

$$(3.3) \qquad \|\phi\|_{W^{s,p}_{\Gamma_0}(\Omega)} \le S_2\|\phi\|_p^{1-s}\|D\phi\|_p^s \le S_2\mu_n(\Omega)^{(1-s)(q-p)/pq}\|\phi\|_q^{1-s}\|D\phi\|_p^s,$$

since $p < q$, and μ_n is the n–dimensional Lebesgue measure on Ω. In conclusion,

$$(3.4) \qquad \|\phi\|_{\wp,\Gamma_1} \le S\|\phi\|_q^{1-s}\|D\phi\|_p^s \quad \text{for all } \phi \in W^{1,p}_{\Gamma_0}(\Omega),$$

where $S = S_0S_1S_2\mu_n(\Omega)^{(1-s)(q-p)/pq}$. Without loss of generality we assume $S \ge 1$, since $s < 1$ and $q > p$.

Suppose that there exists $k \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}_0^+)$, *with* $k' \ge 0$ *in* $\mathbb{R}^+$, $k_0 = k(0) > 0$, *verifying*

$$(3.5) \qquad \delta_1^{1/(m-1)} + \delta_2^{1/(\wp-1)} \le k \quad \text{in } \mathbb{R}_0^+,$$

and

$$(3.6) \qquad \int^\infty k(t)^{-(1+\theta)}dt = \infty,$$

for some $\theta \in (0,\theta_0]$, *where*

$$(3.7) \qquad \theta_0 = \min\left\{\frac{q-2}{q+2}, \frac{\overline{r}}{1-\overline{r}}\right\}, \quad \overline{r} = \frac{1}{\wp} - \left(\frac{1-s}{q} + \frac{s}{p}\right), \quad r = \frac{\theta}{1+\theta}.$$

Put

$$(3.8)\qquad \begin{gathered} \alpha_1 = (1-s)\left(1+\frac{\kappa}{m}\right) - q\left\{\frac{1}{m} - \frac{s}{p}\left(1+\frac{\kappa}{m}\right)\right\},\ \alpha_2 = 1-s-q\left(\frac{1}{\wp}-\frac{s}{p}\right), \\ q_1 = 2^{1/(m-1)} c_1^{\alpha_1-\alpha_2} S^{1+\kappa/m}, \quad \mathfrak{C} = S_1 S_2 \mu_n(\Omega)^{(1-s)(q-p)/pq+(\wp_0-2)/2\wp_0}, \\ \mathcal{C} = \mathfrak{C}^{2/(1-2r)} \left(c_1/\mathfrak{C}_q\right)^{2sq/[q(1-2r)-2(1-s)]-p}, \end{gathered}$$

where c_1 *is given in Proposition* 2.4 *and satisfies* (2.18).

Observe that

$$(3.9)\qquad \frac{n}{p} - \frac{n-1}{\wp} < s < \left(\frac{q}{\wp}-1\right) \Big/ \left(\frac{q}{p}-1\right),$$

being $\wp < \wp_0$. Combining (3.9) with the fact that $\kappa \le (\wp - m)p/\wp$ by (2.8), we get $\alpha_1 \le \alpha_2 < 0$.

From now on, we denote by $u_0(x) = u(0,x)$ *and* $u_1(x) = u_t(0,x)$ *for each* $x \in \Omega$.

THEOREM 3.1. *Assume that*

$$(3.10)\qquad Eu(0) < E_0 \quad \text{and} \quad \|Du(0,\cdot)\|_p > \upsilon_0.$$

Denoted by

$$(3.11)\qquad \begin{gathered} \mathscr{H}_0 = E_0 - [Eu(0)]^+ > 0, \quad 0 < \varepsilon_0 < \min\left\{q-\sigma,\, q-\gamma p - \frac{q[Eu(0)]^+}{w_0}\right\}, \\ c_2 = \min\left\{\frac{\overline{c}\varepsilon_0}{q}, \frac{\varepsilon_0(q-\varepsilon_0-\gamma p)}{pq}\left[a + b\left(\frac{c_1}{\mathfrak{C}_q}\right)^{p(\gamma-1)}\right]\right\} > 0, \\ \ell = \min\left\{\frac{1}{2}, \frac{c_2}{2q_1}\left(\frac{\gamma p \mathscr{H}_0}{c_\infty}\right)^{\overline{r}}\right\}, \\ \lambda = \max\left\{\frac{q_1 (c_\infty/\gamma p)^{\overline{r}} \mathscr{H}_0^{r-\overline{r}}}{(1-r)\ell^{m'/m}}, \frac{2[\langle u_0,u_1\rangle + \langle u_0,u_1\rangle_{\Gamma_1}]^-}{k_0 \mathscr{H}_0^{1/(1+\theta)}}, \frac{1}{k_0}\right\}, \\ \mathcal{K} = 2\cdot 4^{\frac{r}{1-r}}\left\{\frac{c_\infty}{\gamma p} + \mathcal{C} + c_1^{\frac{2-q(1-2r)}{1-2r}} \mu_n(\Omega)^{\frac{q-2}{q(1-2r)}}\right\} \max\left\{1, \frac{1}{c_2}\right\}, \\ \mathscr{Z}_0 = \lambda k_0 \mathscr{H}_0^{1-r} + \langle u_0,u_1\rangle + \langle u_0,u_1\rangle_{\Gamma_1}, \end{gathered}$$

then $T \le T_0$*, where* T_0 *is the unique positive number satisfying*

$$(3.12)\qquad \int_0^{T_0} k(t)^{-(1+\theta)} dt = \frac{\mathcal{K}\lambda}{\theta}\left(\frac{\lambda}{\mathscr{Z}_0}\right)^{\theta}.$$

PROOF. We somehow follow the ideas contained in [**4**, **6**]. For each $t \in I$ put

$$\mathscr{H}(t) = \mathscr{H}_0 + \int_0^t \mathscr{D}u(\tau)d\tau.$$

Of course, $\mathscr{H}$ is well defined and non–decreasing, being $\mathscr{D} \ge 0$ and finite along u. Moreover, by (B)–(ii) we get

$$(3.13)\qquad [Eu(0)]^+ - Eu(t) \ge \mathscr{H}(t) \ge \mathscr{H}_0 \quad \text{for } t \in I.$$

Define furthermore for all $t \in I$ the function

$$\mathscr{Z}(t) = \lambda k(t)\left[\mathscr{H}(t)\right]^{1-r} + \langle u_t(t,\cdot), u(t,\cdot)\rangle + \langle u_t(t,\cdot), u(t,\cdot)\rangle_{\Gamma_1},$$

where $r \in (0,1)$ and $\lambda > 0$ are given in (3.7) and (3.11), respectively. Clearly $\mathscr{L} \in W^{1,1}_{\rm loc}(I)$, so that a.e. in I,

$$\mathscr{L}' = \lambda k(1-r)\mathscr{H}^{-r}\mathscr{H}' + \lambda k'\mathscr{H}^{1-r} + \frac{d}{dt}\Big\{\langle u_t, u\rangle + \langle u_t, u\rangle_{\Gamma_1}\Big\}. \tag{3.14}$$

Our aim is to estimate the terms in the right hand side of (3.14). Taking $\phi = u$ in the *Distribution Identity* (A), we obtain

$$\begin{aligned}
&\frac{d}{dt}\Big\{\langle u_t(t,\cdot), u(t,\cdot)\rangle + \langle u_t(t,\cdot), u(t,\cdot)\rangle_{\Gamma_1}\Big\}\\
&\qquad = \|u_t(t,\cdot)\|_2^2 - \langle\langle Au(t,\cdot), u(t,\cdot)\rangle\rangle + \langle f(t,\cdot,u), u(t,\cdot)\rangle\\
&\qquad\quad - \langle Q(t,\cdot,u(t,\cdot),u_t(t,\cdot)), u(t,\cdot)\rangle_{\Gamma_1} + \|u_t(t,\cdot)\|^2_{2,\Gamma_1}\\
&\qquad \ge [1+(q-\varepsilon_0)/2](\|u_t(t,\cdot)\|_2^2 + \|u_t(t,\cdot)\|^2_{2,\Gamma_1}) + (q-\varepsilon_0)\mathscr{A}u(t)\\
&\qquad\quad - \langle\langle Au(t,\cdot), u(t,\cdot)\rangle\rangle + \langle f(t,\cdot,u(t,\cdot)), u(t,\cdot)\rangle - (q-\varepsilon_0)\mathscr{F}u(t)\\
&\qquad\quad - (q-\varepsilon_0)Eu(t) - \langle Q(t,\cdot,u(t,\cdot),u_t(t,\cdot)), u(t,\cdot)\rangle_{\Gamma_1},
\end{aligned}$$

where ε_0 is any positive number taken as in (3.11). By (1.3), (2.5) and (3.11),

$$\begin{aligned}
&\langle f(t,\cdot,u(t,\cdot)), u(t,\cdot)\rangle - (q-\varepsilon_0)\mathscr{F}u(t)\\
&\qquad = \left(1 - \frac{q-\varepsilon_0}{\sigma}\right)\int_\Omega g(t,x)|u(t,x)|^\sigma dx + \frac{\varepsilon_0}{q}\int_\Omega c(x)|u(t,x)|^q dx\\
&\qquad \ge \frac{\overline{c}\varepsilon_0}{q}\|u(t,\cdot)\|_q^q.
\end{aligned}$$

Therefore, using also (2.11) and (3.13), and recalling that $\varepsilon_0 < q$, we have for all $t \in I$

$$\begin{aligned}
&\frac{d}{dt}\Big\{\langle u_t(t,\cdot), u(t,\cdot)\rangle + \langle u_t(t,\cdot), u(t,\cdot)\rangle_{\Gamma_1}\Big\}\\
&\qquad \ge \|u_t(t,\cdot)\|_2^2 + \|u_t(t,\cdot)\|^2_{2,\Gamma_1} + (\overline{c}\varepsilon_0/q)\|u(t,\cdot)\|_q^q - (q-\varepsilon_0)Eu(t)\\
&\qquad\quad - \langle Q(t,\cdot,u(t,\cdot),u_t(t,\cdot)), u(t,\cdot)\rangle_{\Gamma_1} + (q-\varepsilon_0-\gamma p)\mathscr{A}u(t)\\
&\qquad \ge \|u_t(t,\cdot)\|_2^2 + \|u_t(t,\cdot)\|^2_{2,\Gamma_1} + (\overline{c}\varepsilon_0/q)\|u(t,\cdot)\|_q^q\\
&\qquad\quad + (q-\varepsilon_0-\gamma p)\mathscr{A}u(t) - \langle Q(t,\cdot,u(t,\cdot),u_t(t,\cdot)), u(t,\cdot)\rangle_{\Gamma_1}\\
&\qquad\quad + \gamma p\mathscr{H}(t) - (q-\varepsilon_0)[Eu(0)]^+.
\end{aligned} \tag{3.15}$$

Since $w_1 > w_0$ by Lemma 2.3–(i),

$$\begin{aligned}
(q-\varepsilon_0-\gamma p)\mathscr{A}u(t) - (q-\varepsilon_0)[Eu(0)]^+ &\ge (q-\varepsilon_0-\gamma p)\left(1 - \frac{q-\varepsilon_0}{q}\right)\mathscr{A}u(t)\\
&\quad + (q-\varepsilon_0-\gamma p)\frac{q-\varepsilon_0}{q}w_0 - (q-\varepsilon_0)[Eu(0)]^+\\
&\ge (q-\varepsilon_0-\gamma p)\left(1 - \frac{q-\varepsilon_0}{q}\right)\mathscr{A}u(t),
\end{aligned}$$

being $(q-\varepsilon_0-\gamma p)\dfrac{q-\varepsilon_0}{q}w_0 - (q-\varepsilon_0)[Eu(0)]^+ \ge 0$ thanks to the choice of ε_0 in (3.11). Now, by (2.17) we have

$$(q-\varepsilon_0-\gamma p)\left(1 - \frac{q-\varepsilon_0}{q}\right)\mathscr{A}u(t) \ge C_2\|Du(t,\cdot)\|_p^p,$$

with $C_2 = \varepsilon_0(q-\varepsilon_0-\gamma p)\left[a + b\,(c_1/\mathfrak{C}_q)^{p(\gamma-1)}\right]/pq > 0$. We stress that the positivity of C_2 is guaranteed by the fact that $\varepsilon_0 < q-\gamma p$ in (3.11). Therefore, since $\gamma p\mathscr{H}(t) \geq 0$ for all $t \in I$, from (3.15) we obtain for all $t \in I$

$$\begin{aligned} &\frac{d}{dt}\Big\{\langle u_t(t,\cdot),u(t,\cdot)\rangle + \langle u_t(t,\cdot),u(t,\cdot)\rangle_{\Gamma_1}\Big\} \\ &\qquad \geq \|u_t(t,\cdot)\|_2^2 + \|u_t(t,\cdot)\|_{2,\Gamma_1}^2 + c_2\big(\|u(t,\cdot)\|_q^q + \|Du(t,\cdot)\|_p^p\big) \\ &\qquad - \langle Q(t,\cdot,u(t,\cdot),u_t(t,\cdot)),u(t,\cdot)\rangle_{\Gamma_1}, \end{aligned} \tag{3.16}$$

where $c_2 > 0$ is defined in (3.11). Now, from (2.8) and (3.1) we find

$$\langle Q(t,\cdot,u,u_t),u\rangle_{\Gamma_1} \leq \{\delta_1(t)^{1/m}\|u(t,\cdot)\|_{\wp,\Gamma_1}^{\kappa/m}\mathscr{D}u(t)^{1/m'} + \delta_2(t)^{1/\wp}\mathscr{D}u(t)^{1/\wp'}\}\|u(t,\cdot)\|_{\wp,\Gamma_1},$$

see [**5**, Lemma 4.2], and so, by (3.4)

$$\begin{aligned} \langle Q(t,\cdot,u,u_t),u\rangle_{\Gamma_1} \leq S\{&\delta_1(t)^{1/m}\|u(t,\cdot)\|_{\wp,\Gamma_1}^{\kappa/m}\mathscr{D}u(t)^{1/m'} \\ &+ \delta_2(t)^{1/\wp}\mathscr{D}u(t)^{1/\wp'}\}\|u(t,\cdot)\|_q^{1-s}\|Du(t,\cdot)\|_p^{s}. \end{aligned} \tag{3.17}$$

Let

$$\frac{1}{\beta_1} = \frac{1}{m} - \frac{s}{p}\left(1+\frac{\kappa}{m}\right), \quad \frac{1}{\beta_2} = \frac{1}{\wp} - \frac{s}{p}.$$

We claim that $1 < \beta_1 \leq \beta_2$. Indeed, $\beta_1 > 1$ derives from the facts that $s > 0$, $m+\kappa > 0$ and $m > 1$ by (2.8). On the other hand, the relation $\beta_1 \leq \beta_2$ is equivalent to $s\kappa \leq (\wp - m)p/\wp$, which holds true being $s < 1$ and $\kappa \leq (\wp-m)p/\wp$ by (2.8).

Hence, (3.17) and the fact that $S \geq 1$ imply that for all $t \in I$

$$\begin{aligned} &\langle Q(t,\cdot,u(t,\cdot),u_t(t,\cdot)),u(t,\cdot)\rangle_{\Gamma_1} \\ &\leq S^{1+\kappa/m}\Bigg\{\Big(\delta_1(t)^{1/(m-1)}\mathscr{D}u(t)\Big)^{1/m'}\cdot\|u(t,\cdot)\|_q^{(1-s)(1+\kappa/m)}\|Du(t,\cdot)\|_p^{s(1+\kappa/m)} \\ &\qquad + \Big(\delta_2(t)^{1/(\wp-1)}\mathscr{D}u(t)\Big)^{1/\wp'}\|u(t,\cdot)\|_q^{1-s}\|Du(t,\cdot)\|_p^{s}\Bigg\} \\ &= S^{1+\kappa/m}\Bigg\{\Big(\delta_1(t)^{1/(m-1)}\mathscr{D}u(t)\Big)^{1/m'}\|u(t,\cdot)\|_q^{q/\beta_1}\|Du(t,\cdot)\|_p^{s(1+\kappa/m)}\|u(t,\cdot)\|_q^{\alpha_1} \\ &\qquad + \Big(\delta_2(t)^{1/(\wp-1)}\mathscr{D}u(t)\Big)^{1/\wp'}\|u(t,\cdot)\|_q^{q/\beta_2}\|Du(t,\cdot)\|_p^{s}\|u(t,\cdot)\|_q^{\alpha_2}\Bigg\} \\ &\leq S^{1+\kappa/m}\Big\{[(2\delta_1(t)/\ell)^{1/(m-1)}\mathscr{D}u(t) + \tfrac{1}{2}\ell\|u(t,\cdot)\|_q^q + \tfrac{1}{2}\ell\|Du(t,\cdot)\|_p^p]\cdot\|u(t,\cdot)\|_q^{\alpha_1} \\ &\qquad + [(2\delta_2(t)/\ell)^{1/(\wp-1)}\mathscr{D}u(t) + \tfrac{1}{2}\ell\|u(t,\cdot)\|_q^q + \tfrac{1}{2}\ell\|Du(t,\cdot)\|_p^p]\cdot\|u(t,\cdot)\|_q^{\alpha_2}\Big\}, \end{aligned}$$

where in the last step we have applied Young's inequality, with $\ell \in (0,1)$ given in (3.11). Finally, by $(2.16)_1$

$$\begin{aligned} \langle Q(t,\cdot,u(t,\cdot),u_t(t,\cdot)),u(t,\cdot)\rangle_{\Gamma_1} \leq q_1\Big\{&\ell^{-m'/m}[\delta_1(t)^{1/(m-1)} + \delta_2(t)^{1/(\wp-1)}]\mathscr{D}u(t) \\ &+ \ell\big(\|u(t,\cdot)\|_q^q + \|Du(t,\cdot)\|_p^p\big)\Big\}\cdot\|u(t,\cdot)\|_q^{\alpha_2}, \end{aligned}$$

where $q_1 = 2^{1/(m-1)} S^{1+\kappa/m} \max\{1, c_1^{\alpha_1-\alpha_2}\} > 0$ is exactly the number defined in (3.8), being $c_1 \le 1$. Now, (2.7), (2.10), (3.13) and Lemma 2.3–(iii) assure for all $t \in I$ that

$$\mathscr{H}(t) \le E_0 - Eu(t) < \left(\frac{q}{\gamma p} - 1\right) w_2 + \mathscr{F}u(t) \le \frac{q}{\gamma p}\mathscr{F}u(t) \le \frac{c_\infty}{\gamma p}\|u(t,\cdot)\|_q^q. \tag{3.18}$$

Moreover, by (3.9)

$$\overline{r} = -\alpha_2/q \in (0,1),$$

and so

$$\|u(t,\cdot)\|_q^{\alpha_2} = \|u(t,\cdot)\|_q^{-q\overline{r}} \le (c_\infty/q)^{\overline{r}}[\mathscr{F}u(t)]^{-\overline{r}} \le (c_\infty/\gamma p)^{\overline{r}}[\mathscr{H}(t)]^{-\overline{r}}.$$

Therefore,

$$\begin{aligned}\langle Q(t,\cdot,u(t,\cdot),u_t(t,\cdot)),u(t,\cdot)\rangle_{\Gamma_1}&\\ \le q_1(c_\infty/\gamma p)^{\overline{r}}\Big\{&\ell^{-m'/m}[\delta_1(t)^{1/(m-1)} + \delta_2(t)^{1/(\wp-1)}]\mathscr{D}u(t)\\ &+ \ell\big(\|u(t,\cdot)\|_q^q + \|Du(t,\cdot)\|_p^p\big)\Big\}[\mathscr{H}(t)]^{-\overline{r}}\end{aligned}$$

for all $t \in I$. Put

$$r_0 = \min\left\{\frac{1}{2} - \frac{1}{q}, \overline{r}\right\}. \tag{3.19}$$

Note that θ_0 in (3.7) can be expressed as $\theta_0 = r_0/(1-r_0)$ so that $r \in (0, r_0)$. Consequently, since $0 < r < r_0 \le \overline{r} < 1$ by (3.19) and $\mathscr{H} \ge \mathscr{H}_0$, we have

$$\begin{aligned}&\langle Q(t,\cdot,u(t,\cdot),u_t(t,\cdot)),u(t,\cdot)\rangle_{\Gamma_1}\\ &\quad\le q_1(c_\infty/\gamma p)^{\overline{r}}\Big\{\ell\mathscr{H}_0^{-\overline{r}}\big(\|u(t,\cdot)\|_q^q + \|Du(t,\cdot)\|_p^p\big)\\ &\quad+ \ell^{-m'/m}\mathscr{H}_0^{r-\overline{r}}[\delta_1(t)^{1/(m-1)} + \delta_2(t)^{1/(\wp-1)}]\cdot[\mathscr{H}(t)]^{-r}\mathscr{D}u(t)\Big\}.\end{aligned} \tag{3.20}$$

Therefore, by (3.5), (3.16), (3.20) and the facts that $\lambda k'\mathscr{H}^{1-r} \ge 0$ and $\mathscr{H}' = \mathscr{D}$, from (3.14) it follows that a.e. in I

$$\begin{aligned}\mathscr{L}' \ge k\left\{\lambda(1-r) - q_1(c_\infty/\gamma p)^{\overline{r}}\ell^{-m'/m}\mathscr{H}_0^{r-\overline{r}}\right\}\mathscr{H}^{-r}\mathscr{H}' + \|u_t(t,\cdot)\|_2^2 + \|u_t(t,\cdot)\|_{2,\Gamma_1}^2\\ + \left\{c_2 - q_1(c_\infty/\gamma p)^{\overline{r}}\ell\mathscr{H}_0^{-\overline{r}}\right\}(\|u(t,\cdot)\|_q^q + \|Du(t,\cdot)\|_p^p).\end{aligned}$$

Since $\lambda(1-r) - q_1(c_\infty/\gamma p)^{\overline{r}}\ell^{-m'/m}\mathscr{H}_0^{r-\overline{r}} \ge 0$, for a.a. $t \in I$

$$\mathscr{L}'(t) \ge C\Big\{\|u_t(t,\cdot)\|_2^2 + \|u_t(t,\cdot)\|_{2,\Gamma_1}^2 + \|u(t,\cdot)\|_q^q + \|Du(t,\cdot)\|_p^p\Big\}, \tag{3.21}$$

where $2C = \min\{c_2, 1\} \le 1$. On the other hand, putting $\alpha = 1/(1-r) \in (1,2)$, from the definition of $\mathscr{L}$ we obtain

$$\begin{aligned}\mathscr{L}(t) &\le \lambda k(t)\mathscr{H}(t)^{1/\alpha} + \Big\{|\langle u_t(t,\cdot),u(t,\cdot)\rangle| + |\langle u_t(t,\cdot),u(t,\cdot)\rangle_{\Gamma_1}|\Big\}\\ &\le \lambda k(t)\mathscr{H}(t)^{1/\alpha} + \Big\{\|u_t(t,\cdot)\|_2\|u(t,\cdot)\|_2 + \|u_t(t,\cdot)\|_{2,\Gamma_1}\|u(t,\cdot)\|_{2,\Gamma_1}\Big\}.\end{aligned}$$

Denote by $\nu = 2/\alpha$ so that $\nu > 1$. By Young's inequality

$$\begin{aligned}\mathscr{L}(t)^\alpha \le 4^{\alpha-1}\left[\max\{\lambda k(t),1\}\right]^\alpha\Big\{&\mathscr{H}(t) + \|u_t(t,\cdot)\|_2^{\alpha\nu} + \|u(t,\cdot)\|_2^{\alpha\nu'}\\ &+\|u_t(t,\cdot)\|_{2,\Gamma_1}^{\alpha\nu} + \|u(t,\cdot)\|_{2,\Gamma_1}^{\alpha\nu'}\Big\}.\end{aligned} \tag{3.22}$$

In order to estimate the right hand side of (3.22), first note that $L^q(\Omega) \hookrightarrow L^2(\Omega)$ continuously being $q > 2$, and so, by $(2.16)_1$, we get for all $t \in I$

$$\begin{aligned}\|u(t,\cdot)\|_2^{\alpha\nu'} &\le \mu_n(\Omega)^{\alpha\nu'(q-2)/2q}\|u(t,\cdot)\|_q^{\alpha\nu'} \\ &\le c_1^{\alpha\nu'-q}\mu_n(\Omega)^{\alpha\nu'(q-2)/2q}\|u(t,\cdot)\|_q^q.\end{aligned} \tag{3.23}$$

Consider the relation

$$z^\xi \le z+1 \le (1+1/\eta)(z+\eta),$$

which holds for all $z \ge 0$, $\xi \in [0,1]$, $\eta > 0$. Take $z = \|u(t,\cdot)\|_{2,\Gamma_1}^{\alpha\nu'}$, $\xi = 2/\alpha\nu'$ and $\eta = \mathscr{H}_0$, so that

$$\|u(t,\cdot)\|_{2,\Gamma_1}^2 \le (1+1/\mathscr{H}_0)(\mathscr{H}_0 + \|u(t,\cdot)\|_{2,\Gamma_1}^{\alpha\nu'}) \le (1+1/\mathscr{H}_0)(\mathscr{H}(t) + \|u(t,\cdot)\|_{2,\Gamma_1}^{\alpha\nu'}).$$

By Proposition 2.1, (3.2) and (3.3) we have

$$\|u(t,\cdot)\|_{2,\Gamma_1} \le \mu_n(\Omega)^{(\wp_0-2)/2\wp_0}\|u(t,\cdot)\|_{\wp_0,\Gamma_1} \le \mathfrak{C}\|u(t,\cdot)\|_q^{1-s}\|Du(t,\cdot)\|_p^s,$$

with $\mathfrak{C}$ given in (3.8). Raising both sides of the last relation to the power $\alpha\nu'$ and then using Young's inequality with exponents $\sigma_1 = q/(1-s)\alpha\nu'$ and $\sigma_2 = q/[q-(1-s)\alpha\nu']$, we get

$$\|u(t,\cdot)\|_{2,\Gamma_1}^{\alpha\nu'} \le \mathfrak{C}^{\alpha\nu'}\left(\|u(t,\cdot)\|_q^q + \|Du(t,\cdot)\|_p^{s\alpha\nu'\sigma_2}\right). \tag{3.24}$$

This is possible, since $\sigma_1 > 1$ and $\sigma_2 > 1$. Indeed, $s \in (0,1)$ and $q > \alpha\nu'$ by (3.19), being

$$\frac{1}{\alpha\nu'} = \frac{\nu-1}{\alpha\nu} = \frac{1}{\alpha} - \frac{1}{2} = \frac{1}{2} - r > \frac{1}{q}.$$

We claim that

$$s\alpha\nu'\sigma_2 < p. \tag{3.25}$$

Relation (3.25) is equivalent to

$$\alpha\nu' < pq/[s(q-p)+p]. \tag{3.26}$$

Since $\alpha\nu = 2$, the function $\alpha\nu' = 2\alpha/(2-\alpha)$ is strictly increasing in the variable α. Now $\alpha = 1/(1-r)$ and $r < \overline{r}$ by (3.19), so that

$$\alpha\nu' < \frac{2}{1-2\overline{r}} = \frac{2pq\wp}{pq(\wp-2)+2s\wp(q-p)+2p\wp}.$$

Hence, to prove (3.26) it is sufficient to show that

$$\frac{2\wp}{pq(\wp-2)+2s\wp(q-p)+2p\wp} \le \frac{1}{s(q-p)+p},$$

which clearly holds, being $\wp \ge 2$. Therefore, the claim (3.25) is true and from (3.24) we get

$$\|u(t,\cdot)\|_{2,\Gamma_1}^{\alpha\nu'} \le \mathcal{C}\left(\|u(t,\cdot)\|_q^q + \|Du(t,\cdot)\|_p^p\right),$$

by (2.16), where $\mathcal{C} = \mathfrak{C}^{\alpha\nu'}\max\{1,(c_1/\mathfrak{C}_q)^{s\alpha\nu'\sigma_2-p}\} = \mathfrak{C}^{\alpha\nu'}(c_1/\mathfrak{C}_q)^{s\alpha\nu'\sigma_2-p} \ge 1$ is exactly the positive number given in (3.8). Consequently, by (3.18) and (3.23) it follows

$$\begin{aligned}\mathscr{Z}(t)^\alpha &\le 4^{\alpha-1}\left[\max\{\lambda k(t),1\}\right]^\alpha \left\{\|u_t(t,\cdot)\|_2^2+\|u_t(t,\cdot)\|_{2,\Gamma_1}^2\right.\\ &\quad\left.+\left(c_\infty/\gamma p+\mathcal{C}+c_1^{\alpha\nu'-q}\mu_n(\Omega)^{\alpha\nu'(q-2)/2q}\right)\|u(t,\cdot)\|_q^q+\mathcal{C}\|Du(t,\cdot)\|_p^p\right\}\\ &\le 4^{\alpha-1}\left[\max\{\lambda k(t),k(t)/k_0\}\right]^\alpha \left\{\|u_t(t,\cdot)\|_2^2+\|u_t(t,\cdot)\|_{2,\Gamma_1}^2\right.\\ &\quad\left.+\left(c_\infty/\gamma p+\mathcal{C}+c_1^{\alpha\nu'-q}\mu_n(\Omega)^{\alpha\nu'(q-2)/2q}\right)\|u(t,\cdot)\|_q^q+\mathcal{C}\|Du(t,\cdot)\|_p^p\right\},\end{aligned}$$

being $k(t)\ge k_0>0$ for a.a. $t\in I$. Since $\lambda\ge 1/k_0$ by assumption, taking $B=4^{\alpha-1}(c_\infty/\gamma p+\mathcal{C}+c_1^{\alpha\nu'-q}\mu_n(\Omega)^{\alpha\nu'(q-2)/2q})$, we obtain

$$\mathscr{Z}(t)^\alpha\le B[\lambda k(t)]^\alpha\left\{\|u_t(t,\cdot)\|_2^2+\|u_t(t,\cdot)\|_{2,\Gamma_1}^2+\|u(t,\cdot)\|_q^q+\|Du(t,\cdot)\|_p^p\right\}. \tag{3.27}$$

Combining the last relation with (3.21), we have

$$\mathscr{Z}(t)^{-\alpha}\mathscr{Z}'(t)\ge\frac{C}{B}[\lambda k(t)]^{-\alpha}.$$

In conclusion, for a.a. $t\in I$

$$\mathscr{Z}(t)^\theta\ge\frac{B\lambda^{1+\theta}}{B\lambda^{1+\theta}\mathscr{Z}_0^{-\theta}-\theta C\int_0^t k(\tau)^{-(1+\theta)}d\tau}=\Phi(t), \tag{3.28}$$

where $\mathscr{Z}_0=\mathscr{Z}(0)$. Therefore, $\Phi(t)\nearrow\infty$ as $t\nearrow T_0$, where T_0 is defined in (3.12) and the constant $\mathcal{K}$ given in (3.11) is obtained as $\mathcal{K}=B/C$. Hence $\mathscr{Z}$ cannot be continued after T_0, that is u cannot be global and $T\le T_0$, as required. □

Remark 3.2. (*i*) The request $\lambda\ge 2[\langle u_0,u_1\rangle+\langle u_0,u_1\rangle_{\Gamma_1}]^-/k_0\mathscr{H}_0^{1/(1+\theta)}$ in (3.11) guarantees that $\mathscr{Z}_0>0$, in the more subtle case $\langle u_0,u_1\rangle+\langle u_0,u_1\rangle_{\Gamma_1}<0$. In the literature, when the initial data u_0 and u_1 are such that $\langle u_0,u_1\rangle\ge 0$, they are called *cooperative*. In this context we generalize this notion, saying that u_0 and u_1 are *cooperative up to the boundary* if $\langle u_0,u_1\rangle\ge 0$ and $\langle u_0,u_1\rangle_{\Gamma_1}\ge 0$. If u_0 and u_1 are *cooperative up to the boundary* then $\mathscr{Z}_0>0$, being $\lambda>0$ by (3.11), and in this case condition (3.11) on λ simply reduces to

$$\lambda=\max\left\{\frac{q_1(c_\infty/\gamma p)^{\overline{r}}\mathscr{H}_0^{r-\overline{r}}}{(1-r)\ell^{m'/m}},\frac{1}{k_0}\right\}.$$

(*ii*) If either $Eu(0)>0$ or $Eu(0)\le 0$ *and* $\sigma>\gamma p$, then we can take

$$\varepsilon_0=\min\left\{q-\sigma,\,q-\gamma p-\frac{q[Eu(0)]^+}{w_0}\right\}\in(0,q-\gamma p)$$

in (3.11). In condition (6.4) of [**4**], a similar request on ε_0, with 2 in place of p, was made in order to obtain a priori estimates for polyharmonic Kirchhoff systems under homogeneous Dirichlet boundary conditions. However, in [**4**] the possibility $\varepsilon_0=q-2\gamma$ was allowed, while here we strongly need $\varepsilon_0<q-\gamma p$, as stressed in the proof of Theorem 3.1.

(*iii*) Theorem 3.1 does not guarantee finite time blow up of solutions. However, global non–existence occurs by the blow up of natural norms, when either $T=T_0$ or $\lim_{t\to T^-}\mathscr{Z}(t)=\infty$, as it will be shown in the corollary below.

Corollary 3.3. *Under the assumptions of Theorem* 3.1, *if either* $\lim_{t\to T^-}\mathscr{Z}(t)=\infty$ *or* $T=T_0$, *then*

$$\lim_{t\to T^-}\|Du(t,\cdot)\|_p=\infty. \tag{3.29}$$

Proof. The proof of Theorem 3.1 can be repeated word by word. Hence, by (3.28) we get $\lim_{t\to T^-}\mathscr{Z}(t)=\infty$ in both cases. Now, relations (2.10), (3.13) and (B)–(*ii*) imply that for all $t\in I$

$$0<\mathscr{H}_0\le\mathscr{H}(t)\le[Eu(0)]^+-Eu(t)\le[Eu(0)]^+-\tfrac{1}{2}\left(\|u_t(t,\cdot)\|_2^2+\|u_t(t,\cdot)\|_{2,\Gamma_1}^2\right)+\mathscr{F}u(t).$$

Hence, by Lemma 2.3–(*iii*) and (2.7)

$$\begin{aligned}\|u_t(t,\cdot)\|_2^2+\|u_t(t,\cdot)\|_{2,\Gamma_1}^2&<2([Eu(0)]^++\mathscr{F}u(t))<2(E_0+\mathscr{F}u(t))\\&\le\frac{2q}{\gamma p}\mathscr{F}u(t)\le\frac{2c_\infty}{\gamma p}\|u(t,\cdot)\|_q^q.\end{aligned}$$

Using also (2.2) and (2.16), we get

$$\mathscr{Z}(t)^\alpha\le B[\lambda k(t)]^\alpha\left\{2\mathscr{F}u(t)+2E_0+\|u(t,\cdot)\|_q^q+\|Du\|_p^p\right\}\le\Lambda\|Du(t,\cdot)\|_p^q,$$

where α and B are the positive constants introduced in the proof of Theorem 3.1, and $\Lambda=B[\lambda k(T)]^\alpha\left\{\left(\dfrac{2c_\infty}{\gamma p}+1\right)\mathfrak{C}_q^q+\left(\dfrac{\mathfrak{C}_q}{c_1}\right)^{q-p}\right\}>0$ is obtained by the monotonicity of k, being $T\le T_0<\infty$. Therefore,

$$\|Du(t,\cdot)\|_p^q\ge\Lambda^{-1}\mathscr{Z}(t)^\alpha,$$

and so $\lim_{t\to T^-}\|Du(t,\cdot)\|_p=\infty$, as claimed. □

Of course there exists $\lim_{t\to T^-}\mathscr{Z}(t)\le\infty$ by (3.22) and (3.28). If $\lim_{t\to T^-}\mathscr{Z}(t)$ is infinite, a case which occurs when $T=T_0$, then (3.29) is valid as shown in Corollary 3.3. While, if $\lim_{t\to T^-}\mathscr{Z}(t)=\mathscr{Z}_T<\infty$, so that $T<T_0$ by Corollary 3.3, it could happen that $\limsup_{t\to T^-}\|Du(t,\cdot)\|_p<\infty$, as explained in Remark 3.2–(*iii*). In this case, or even when $\liminf_{t\to T^-}\|Du(t,\cdot)\|_p$ is finite, we get $\lim_{t\to T^-}\mathscr{H}(t)<\infty$. Otherwise, by the definition of $\mathscr{H}$ there exists $\lim_{t\to T^-}\mathscr{H}(t)=\infty$ and so $\lim_{t\to T^-}\|u(t,\cdot)\|_q=\infty$ by (3.18). This is clearly impossible by the Sobolev imbedding, being q subcritical by (2.5). Therefore, the main dynamical part $\mathscr{D}u$ of the damped system, the so called *damping rate*, is actually in $L^1(I)$, $I=[0,T)$, and this means that *the total damping over the entire time interval I is finite.*

In the next Corollary 3.4, we give simpler expressions for T_0, when the damping Q is of special type. We assume all the structural hypothesis stated at the beginning of the Section, except for the existence of the auxiliary function k satisfying (3.5) and (3.6). The proof of Corollary 3.4 will consist essentially in finding such a function. To this aim, we somehow follow the proof of Proposition 4.1 of [**4**], writing all the steps for more clarity.

Corollary 3.4. *Given $\mathfrak{K}\ge1$, $0\le\mathfrak{s}\le m-1$ and δ_1, δ_2 defined in* (3.1), *suppose that*

$$\delta_1(t)^{1/(m-1)}+\delta_2(t)^{1/(\wp-1)}\le\mathfrak{K}(1+t)^{\mathfrak{s}/(m-1)}\quad\textit{for all } t\in I.$$

If (3.10) *holds then $T\le T_0$ with*

$$T_0=\begin{cases}\dfrac{\mathcal{K}(\lambda\mathfrak{K})^{1+\theta}}{\theta\mathscr{Z}_0^\theta}=\Theta_0, & \textit{if }\ \mathfrak{s}=0\ \textit{ or }\ \mathfrak{s}=m-1,\quad 0<\theta\le\theta_0,\\ e^{\Theta_0}-1, & \textit{if }\ 0<\mathfrak{s}<m-1,\quad \theta=(m-1-\mathfrak{s})/\mathfrak{s},\\ (\mathfrak{m}\Theta_0+1)^{1/\mathfrak{m}}-1, & \textit{if }\ 0<\mathfrak{s}<m-1,\quad \theta<(m-1-\mathfrak{s})/\mathfrak{s},\end{cases}$$

where $\mathfrak{m} = [m - 1 - \mathfrak{s}(1+\theta)]/(m-1) > 0$ *being* $\theta < (m-1-\mathfrak{s})/\mathfrak{s}$*, and* λ*,* $\mathcal{K}$ *and* $\mathscr{Z}_0$ *are the positive constants defined in* (3.11)*.*

Proof. First we need to find a function $k \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^+_0)$, $k > 0$, $k' \geq 0$ and a positive number θ satisfying (3.5) and (3.6). Define for all $t \in \mathbb{R}^+_0$

$$k(t) = \begin{cases} \mathfrak{K}(1+t)^{\mathfrak{s}/(m-1)}, & \text{if } 0 \leq \mathfrak{s} < m-1, \\ \mathfrak{K}, & \text{if } \mathfrak{s} = m-1. \end{cases}$$

In both the cases $k \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^+_0)$, $k > 0$, $k' \geq 0$ and (3.5) holds.

Moreover, if $\mathfrak{s} = 0$ or $\mathfrak{s} = m-1$, then (3.6) holds taking any $\theta \in (0, \theta_0]$, with θ_0 as in (3.7), and the value $\theta = \theta_0$ is optimal. While, if $0 < \mathfrak{s} < m-1$, then (3.6) holds, provided that $\theta > 0$ is so small that $\theta \leq \min\{\theta_0, (m-1-\mathfrak{s})/\mathfrak{s}\}$.

To conclude the proof it is enough to apply Theorem 3.1, so that from (3.12) we get the claim. □

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